WAmaths Mathematics Methods 12

Chapter 1 – Differentiation

EXERCISE 1.1 Differentiating simple functions

Question 1

a Write in the form $f(x) = ax^n$: $f(x) = -3x^6$

Differentiate using $f'(x) = a \times nx^{n-1}$.

$f'(x) = -3 \times 6x^{6-1} = \mathbf{-18x^5}$

b Write in the form $f(x) = ax^n$:

$$f(x) = \frac{4}{5x^{\frac{5}{4}}} = \frac{4}{5}x^{-\frac{5}{4}}$$

Differentiate using $f'(x) = a \times nx^{n-1}$.

$$f'(x) = \frac{4}{5} \times -\frac{5}{4}x^{-\frac{5}{4}-1}$$

$$= -x^{-\frac{9}{4}}$$

$$\mathbf{= -\frac{1}{x^{\frac{9}{4}}}}$$

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c Write in the form $f(x) = ax^n$:

$$f(x) = x^3 + 3\sqrt{5x^{\frac{2}{3}}}$$

$$= x^3 + 3\left(5x^{\frac{2}{3}}\right)^{\frac{1}{2}}$$

$$= x^3 + 3\sqrt{5}x^{\frac{1}{3}}$$

Differentiate using $f'(x) = a \times nx^{n-1}$.

$$f'(x) = 3x^2 + 3\sqrt{5} \times \frac{1}{3}x^{\frac{1}{3}-1}$$

$$= 3x^2 + \sqrt{5}x^{-\frac{2}{3}}$$

$$= \mathbf{3x^2 + \frac{\sqrt{5}}{x^{\frac{2}{3}}}}$$

Expressed in terms of radicals.

$$= \mathbf{3x^2 + \frac{\sqrt{5}}{\sqrt[3]{x^2}}}$$

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Question 2

a $f(x) = 2x^3 - x^2 - 3$

Differentiate each term.

$\boldsymbol{f'(x) = 6x^2 - 2x}$

Evaluate $f'(-2)$

$$f'(-2) = 6(-2)^2 - 2(-2)$$
$$= 24 + 4$$
$$= \mathbf{28}$$

b Write $f(x)$ in polynomial form.

$$f(x) = \frac{(x-1)(x^2+x+1)}{x-1}$$
$$= x^2 + x + 1$$

Differentiate each term.

$\boldsymbol{f'(x) = 2x + 1}$

Evaluate $f'(2)$

$$f'(2) = 2(2) + 1$$
$$= 4 + 1$$
$$= \mathbf{5}$$

c Write $f(x)$ in polynomial form.

$$f(x) = \frac{6x^{\frac{5}{3}} + 9x^{\frac{11}{3}}}{3x^{\frac{2}{3}}}$$
$$= 2x^{\frac{5}{3}-\frac{2}{3}} + 3x^{\frac{11}{3}-\frac{2}{3}}$$
$$= 2x + 3x^3$$

Differentiate each term.

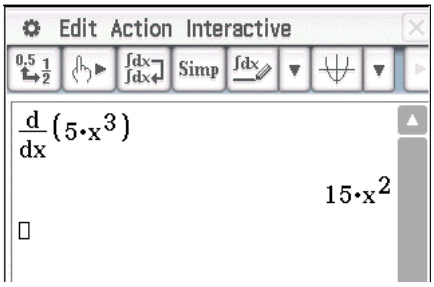
$\boldsymbol{f'(x) = 2 + 9x^2}$

Evaluate $f'(x)$.

$$f'\left(\frac{1}{8}\right) = 2 + 9\left(\frac{1}{8}\right)^2$$
$$= 2 + 9\left(\frac{1}{64}\right)$$
$$= \frac{137}{64}$$
$$= \mathbf{2\frac{9}{64}}$$

Question 3

ClassPad



1 In **Main** enter and highlight the expression $5x^3$.

2 Tap **Interactive** > **Calculation** > **diff**.

3 In the dialogue box, keep the default **Variable: x** and the **Order: 1**.

The answer is $\mathbf{15x^2}$.

TI-Nspire

1 Press **menu** > **Calculus** > **Derivative**.

2 In the derivative template, enter variable **x** in the denominator.

3 Enter the expression $5x^3$ in the brackets as shown above.

4 Press =.

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Question 4

(✓ = 1 mark)

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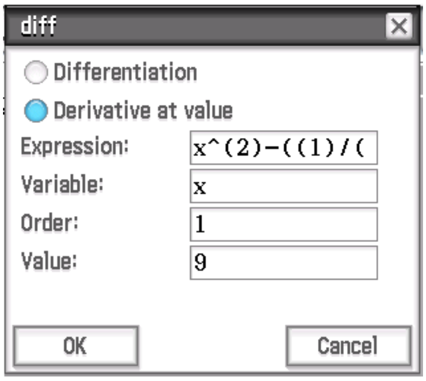


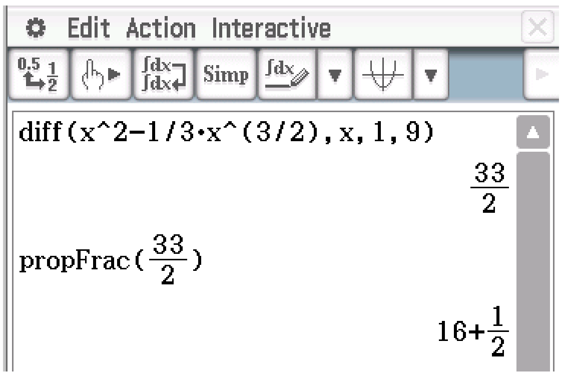
1 In **Main**, enter and highlight the expression $x^2 - \frac{1}{3}x^{\frac{3}{2}}$

2 Tap **Interactive > Calculation > diff**.

3 In the dialogue box, tap **Derivative at value**.

4 Keep the default **Variable: x** and the **Order: 1**.

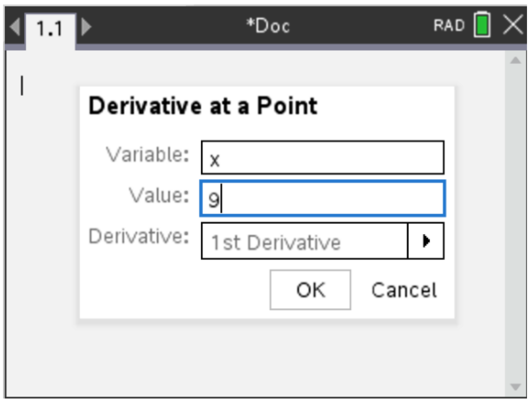
5 In the **Value**: field, enter **9**.



6 The exact value of the derivative will be displayed.

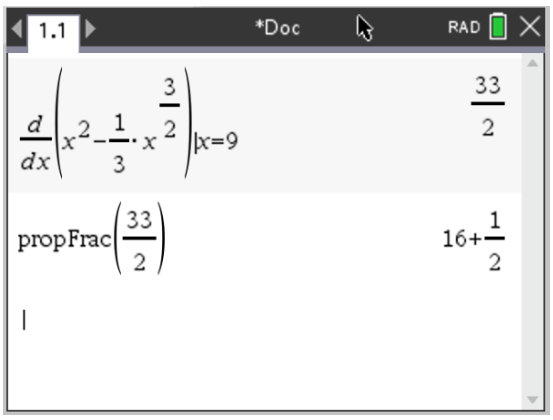
7 Change to **Fraction** mode or use the **Convert** tool for the proper fraction.

ClassPad



1 Press **menu > Calculus > Derivative at a Point**.

2 In the dialogue box **Value**: field, enter **9**.



3 Enter the expression inside the brackets.

4 Press **enter** for the exact solution and

Menu > Number > Fraction tools > Proper Fraction for the proper fraction.

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Question 5

a Write $f(x)$ in polynomial form.

$$f(x)=\frac{x+x^{\frac{2}{3}}+x^{\frac{3}{4}}}{x^{\frac{1}{2}}}$$

$$=x^{\frac{1}{2}}+x^{\frac{2}{3}-\frac{1}{2}}+x^{\frac{3}{4}-\frac{1}{2}}$$

$$=x^{\frac{1}{2}}+x^{\frac{1}{6}}+x^{\frac{1}{4}}$$

Differentiate each term.

$$f'(x)=\frac{1}{2}x^{\frac{1}{2}-1}+\frac{1}{6}x^{\frac{1}{6}-1}+\frac{1}{4}x^{\frac{1}{4}-1}$$

$$=\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{6}x^{-\frac{5}{6}}+\frac{1}{4}x^{-\frac{3}{4}}$$

Therefore, $\boldsymbol{f'(x)=\frac{\frac{1}{2}}{x^{\frac{1}{2}}}+\frac{\frac{1}{6}}{x^{\frac{5}{6}}}+\frac{\frac{1}{4}}{x^{\frac{3}{4}}}}$

correctly find one of the constants✓

correctly find all the constants✓

b

$$f'(2^{12})=\frac{\frac{1}{2}}{2^{12\left(\frac{1}{2}\right)}}+\frac{\frac{1}{6}}{2^{12\left(\frac{5}{6}\right)}}+\frac{\frac{1}{4}}{2^{12\left(\frac{3}{4}\right)}}$$

$$=\frac{\frac{1}{2}}{2^{6}}+\frac{\frac{1}{6}}{2^{10}}+\frac{\frac{1}{4}}{2^{9}}$$

$$=\frac{1}{2^{7}}+\frac{1}{3\times 2^{11}}+\frac{1}{2^{11}}$$

$$=\boldsymbol{\frac{13}{3\times 2^{9}}}\ ✓$$

Question 6 [SCSA MM2021 Q1a] (3 marks)

(✓ = 1 mark)

$$\frac{d}{dx}\left(\frac{3x+1}{x^3}\right)=\frac{(3)x^3-(3x+1)\times 3x^2}{x^6}$$

$$=\frac{3x^3-9x^3-3x^2}{x^6}$$

$$=\frac{-6x^3-3x^2}{x^6}$$

$$=\frac{-6x-3}{x^4}$$

$$=\frac{-3(2x+1)}{x^4}$$

recognises the need for the quotient rule✓

correctly differentiate the expression✓

simplifies the result✓

Question 7 (3 marks)

(✓ = 1 mark)

Given that $f(x)=\frac{1}{a}x^a+a$ and $f'(4) = 16$.

$$f'(x)=\frac{a}{a}x^{a-1}$$

$= \boldsymbol{x^{a-1}}$**✓**

Since $f'(4) = 16$

$f'(4) = 4^{a-1} = 16$✓

$4^{a-1} = 16$

$4^{a-1} = 4^2$

$a - 1 = 2$

$a = 3$✓

Question 8 (2 marks)

(✓ = 1 mark)

$f(x) = \left(\sqrt{3x}\right)^2 - \left(\sqrt{5x^3}\right)^2$

$= 3x - 5x^3$ ✓

The derivative of $f(x)$ is $3 - 15x^2$.✓

Question 9 (2 marks)

(✓ = 1 mark)

$f(x) = \frac{1}{3}x^3 + ax^2 + bx + 1$

Differentiating $f(x)$, we get

$f'(x) = \frac{1}{3} \times 3x^2 + 2ax + b$

$= x^2 + 2ax + b$

It has turning point at $x = -1$, means $f'(-1) = 0$

$f'(-1) = 1 + 2a(-1) + b = 0$

$1 - 2a + b = 0$

$1 = 2a - b$ …[1]

Also, given $f(1) = -2\frac{2}{3}$

$-2\frac{2}{3} = \frac{1}{3} + a + b + 1$

$-2\frac{2}{3} = \frac{4}{3} + a + b$

$-4 = a + b$ …[2]

Solving both equations, we get

$\boldsymbol{a = -1}$ ✓ and $\boldsymbol{b = -3}$ ✓

Question 10 (2 marks)

(✓ = 1 mark)

$f(x) = \frac{1}{2}x^{-2}$

The derivative of $f(x)$ is

$$f'(x) = \frac{-2}{2}x^{-2-1}$$
$$= -x^{-3}$$
$$= -\frac{1}{x^3}$$

$$f'(x+1) = -\frac{1}{(x+1)^3} = \mathbf{-(x+1)^{-3}}$$ ✓

$$f'(x-1) = -\frac{1}{(x-1)^3} = \mathbf{-(x-1)^{-3}}$$ ✓

Question 11 (2 marks)

(✓ = 1 mark)

a $f(x) = 4x^3 + 5x - 9$

$f'(x) = \mathbf{12x^2 + 5}$✓

b **Since $x^2 \geq 0$, for all x.**

Therefore, $12x^2 + 5 \geq 5$ for all x.

$f'(x) \geq 5$ for all x.✓

Question 12 (3 marks)

(✓= 1 mark)

$f(x) = -x^2 + ax$

$f'(x) = a - 2x$✓

$f'(a) = -a$✓

Hence, $g(x) = -ax + c$.

$g(a) = 0$

$\Rightarrow -a \times a + c = 0$

$\mathbf{c = a^2}$✓

Question 13 (1 mark)

(✓ = 1 mark)

$$f(x) = 1 - x + \frac{1}{3}x^3$$

The derivative of $f(x)$ is $f'(x) = -1 + \frac{3}{3}x^2$.

$\boldsymbol{f'(x) = x^2 - 1}$ ✓

Question 14 (1 mark)

(✓ = 1 mark)

$$f(x) = 2x^{\frac{1}{2}} - \frac{2}{3}x^{\frac{3}{2}}$$

Find the derivative of the function $f(x)$.

$$\begin{aligned} f'(x) &= 2 \times \frac{1}{2}x^{\frac{1}{2}-1} - \frac{2}{3} \times \frac{3}{2}x^{\frac{3}{2}-1} \\ &= x^{-\frac{1}{2}} - x^{\frac{1}{2}} \\ &= \frac{1}{x^{\frac{1}{2}}} - \sqrt{x} \\ &= \frac{1}{\sqrt{x}} - \sqrt{x} \\ &= \boldsymbol{\frac{1-x}{\sqrt{x}}} \checkmark \end{aligned}$$

EXERCISE 1.2 The product rule

Question 1

The derivative of $f(x)$ is

$$f'(x)=\frac{3}{3}x^{3-1}-\frac{2}{2}x^{2-1}$$
$$=x^2-x$$

Hence, $f'(a)=a^2-a=a(a-1)$

Therefore, the correct response is **D**.

Question 2

$$f(x)=x^{\frac{1}{2}}+x^{-\frac{1}{2}}$$
$$f'(x)=\frac{1}{2}x^{\frac{1}{2}-1}-\frac{1}{2}x^{-\frac{1}{2}-1}$$
$$=\frac{1}{2}x^{-\frac{1}{2}}-\frac{1}{2}x^{-\frac{3}{2}}$$
$$=\frac{1}{2}\frac{1}{\sqrt{x}}-\frac{1}{2}\frac{1}{x\sqrt{x}}$$
$$=\frac{1}{2\sqrt{x}}\left(1-\frac{1}{x}\right)$$

Therefore, the correct response is **B**.

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Question 3

Use the product rule to differentiate the function $f(x)$.

Identify u and v.

Let $u = 4x + 3x^2$ and $v = 7x^2 - 1$.

$u' = 4 + 6x$ and $v' = 14x$

Write down the expression for $uv' + vu'$.

$f'(x) = (4x + 3x^2) \times 14x + (7x^2 - 1)(4 + 6x)$

Expand and simplify.

$f'(x) = 56x^2 + 42x^3 + 28x^2 + 42x^3 - 4 - 6x$

$= \mathbf{84x^3 + 84x^2 - 6x - 4}$

Question 4

a Use the product rule to differentiate the function $f(x) = x^4(3x + 1)$.

Identify u and v.

Let $u = x^4$ and $v = 3x + 1$.

$u' = 4x^3$ and $v' = 3$

Write down the expression for $uv' + vu'$.

$f'(x) = x^4(3) + (3x + 1) \times 4x^3$

Expand and simplify.

$f'(x) = 3x^4 + 12x^4 + 4x^3$

$= \mathbf{15x^4 + 4x^3}$

b Use the product rule to differentiate the function $f(x) = (4x + 3)(3x - 2)$.

Identify u and v.

Let $u = 4x + 3$ and $v = 3x - 2$.

$u' = 4$ and $v' = 3$

Write down the expression for $uv' + vu'$.

$f'(x) = (4x + 3) \times 3 + (3x - 2) \times 4$

Expand and simplify.

$f'(x) = 12x + 9 + 12x - 8$

$= \mathbf{24x + 1}$

c Use the product rule to differentiate the function $f(x) = 7x(8x - 5)$.

Identify u and v.

Let $u = 7x$ and $v = 8x - 5$.

$u' = 7$ and $v' = 8$

Write down the expression for $uv' + vu'$.

$f'(x) = 7x \times 8 + (8x - 5) \times 7$

Expand and simplify.

$f'(x) = 56x + 56x - 35$

$= \mathbf{112x - 35}$

d Use the product rule to differentiate the function $f(x) = -x^5 (4 - x^2)$.

Identify u and v.

Let $u = -x^5$ and $v = 4 - x^2$.

$u' = -5x^4$ and $v' = -2x$

Write down the expression for $uv' + vu'$.

$f'(x) = -x^5 \times (-2x) + (4 - x^2)(-5x^4)$

Expand and simplify.

$f'(x) = 2x^6 - 20x^4 + 5x^6$

$= \mathbf{7x^6 - 20x^4}$

e Use the product rule to differentiate the function $f(x) = 4x(x^5 - x^2)$.

Identify u and v.

Let $u = 4x$ and $v = x^5 - x^2$.

$u' = 4$ and $v' = 5x^4 - 2x$

Write down the expression for $uv' + vu'$.

Expand and simplify.

$f'(x) = 4x(5x^4 - 2x) + (x^5 - x^2) \times 4$

$= 20x^5 - 8x^2 + 4x^5 - 4x^2$

$= \mathbf{24x^5 - 12x^2}$

f Use the product rule to differentiate the function $f(x) = (5x - 7)(5x + 7)$.

Identify u and v.

Let $u = 5x - 7$ and $v = 5x + 7$.

$u' = 5$ and $v' = 5$

Write down the expression for $uv' + vu'$.

$f'(x) = (5x - 7)5 + (5x + 7)5$

Expand and simplify.

$f'(x) = 25x - 35 + 25x + 35$

$= \mathbf{50x}$

or

$f(x) = (5x - 7)(5x + 7) = 25x^2 - 49$

$f'(x) = 25(2)x$

$= \mathbf{50x}$

g Use the product rule to differentiate the function $f(x) = (1 + 3x)(x^2 - 1)$.

Identify u and v.

Let $u = 1 + 3x$ and $v = x^2 - 1$.

$u' = 3$ and $v' = 2x$

Write down the expression for $uv' + vu'$.

$f'(x) = (1 + 3x) \times 2x + (x^2 - 1) \times 3$

Expand and simplify.

$f'(x) = 2x + 6x^2 + 3x^2 - 3$

$= \mathbf{9x^2 + 2x - 3}$

h Use the product rule to differentiate the function $f(x) = (4x + 5)(2x^3 - 2x + 1)$.

Identify u and v.

Let $u = 4x + 5$ and $v = 2x^3 - 2x + 1$.

$u' = 4$ and $v' = 6x^2 - 2$

Write down the expression for $uv' + vu'$.

$f'(x) = (4x + 5)(6x^2 - 2) + (2x^3 - 2x + 1) \times 4$

Expand and simplify.

$f'(x) = 24x^3 - 8x + 30x^2 - 10 + 8x^3 - 8x + 4$

$= \mathbf{32x^3 + 30x^2 - 16x - 6}$

i Use the chain rule to differentiate the function $f(x) = (x^2 + 1)^2$.

Identify u and v.

Let $u = x^2 + 1$ and $v = x^2 + 1$.

$u' = 2x$ and $v' = 2x$

Write down the expression for $uv' + vu'$.

$f'(x) = (x^2 + 1) \times 2x + (x^2 + 1) \times 2x$

Expand and simplify.

$f'(x) = 2(x^2 + 1) \times 2x$

$= 4x(x^2 + 1)$

$= \mathbf{4x^3 + 4x}$

or

$f(x) = (x^2 + 1)^2$

$= x^4 + 2x^2 + 1$

$f'(x) = \mathbf{4x^3 + 4x}$

Question 5

Use the product rule to differentiate the function $f(x) = (1 + 3x - 2x^3)(6 + x^3 - x^5)$.

Identify u and v.

Let $u = 1 + 3x - 2x^3$ and $v = 6 + x^3 - x^5$.

$u' = 3 - 6x^2$ and $v' = 3x^2 - 5x^4$

Write down the expression for $uv' + vu'$.

$f'(x) = (1 + 3x - 2x^3)(3x^2 - 5x^4) + (6 + x^3 - x^5)(3 - 6x^2)$

Substitute the value and simplify.

Therefore, $f'(-1) = (1 + 3(-1) - 2(-1))(3(1) - 5(1)) + (6 + (-1) - (-1))(3 - 6(1))$

$= 0 + 6(-3)$

$= \mathbf{-18}$✓

Question 6

a Use the product rule to differentiate the function $f(x) = (x^4 + 1)(2x^3 + 5) + (3x^2 - 4)(2x^2 + 5)$.

Identify u and v.

Let $u = x^4 + 1$ and $v = 2x^3 + 5$.

$u' = 4x^3$ and $v' = 6x^2$

Again, let $p = 3x^2 - 4$ and $q = 2x^2 + 5$.

$p' = 6x$ and $q' = 4x$

Write down the expression for $uv' + vu'$ and $pq' + qp'$.

$f'(x) = (x^4 + 1)(6x^2) + (2x^3 + 5)(4x^3) + (3x^2 - 4)(4x) + (2x^2 + 5)(6x)$

Expand and simplify.

$f'(x) = 6x^6 + 6x^2 + 8x^6 + 20x^3 + 12x^3 - 16x + 12x^3 + 30x$

$= \mathbf{14x^6 + 44x^3 + 6x^2 + 14x}$

b Substitute the value of x in the obtained equation.

$f'(1) = 14 + 44(-1) + 6 + 14(-1)$

$= 14 - 44 + 6 - 14$

$= \mathbf{-38}$

Question 7

$f(x) = x^2 + x + 2$

$f'(x) = 2x + 1$

$g(x) = x - 3$

$g'(x) = 1$

$f'(x)g(x) + f(x)g'(x) = (2x + 1)(x - 3) + (x^2 + x + 2) \times 1$

$= 2x^2 - 5x - 3 + x^2 + x + 2$

$= 3x^2 - 4x - 1$

Substitute $x = 3$

$3x^2 - 4x - 1 = 3(3)^2 - 4(3) - 1 = 14$

$\mathbf{f'(3) = 14}$

Question 8

a $f(x) = 2x - 1$

$f'(x) = 2$

$g(x) = x - 3$

$g'(x) = 1$

$f'(x)g(x) + f(x)g'(x) = 2 \times (x - 3) + (2x - 1) \times 1$

$= 4x - 7$

At $x = 2$, gradient is $4(2) - 7 = 1$

$f(2) = (3)(-1) = -3$, so use $(2, -3)$ to find the equation of the tangent line.

$y - (-3) = 1(x - 2)$

The equation of the tangent line is $\boldsymbol{y = x - 5}$

b Stationary points when derivative of $f(x) = (2x - 1)(x - 3)$ is zero.

$4x - 7 = 0$ so $x = \dfrac{7}{4}$

$$f\left(\frac{7}{4}\right) = \left(2 \times \frac{7}{4} - 1\right)\left(\frac{7}{4} - 3\right) = \left(-\frac{25}{8}\right)$$

Stationary point at $\boldsymbol{\left(\frac{7}{4}, -\frac{25}{8}\right)}$

Question 9

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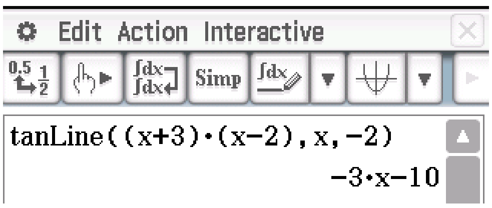

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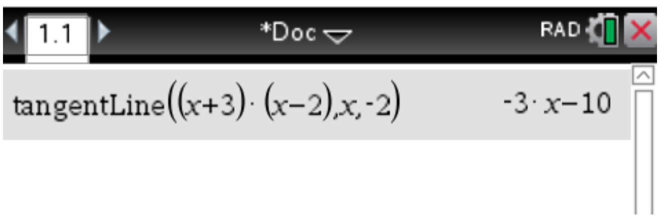

$\boldsymbol{y = -3x - 10}$

Question 10 (1 mark)

(✓= 1 mark)

$$y = x^{\frac{5}{2}} + x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{5}{2}x^{\frac{3}{2}} + \frac{1}{2}x^{-\frac{1}{2}}$$

For $x = 4$,

$$\frac{dy}{dx} = \frac{5}{2}(4)^{\frac{3}{2}} + \frac{1}{2}(4)^{-\frac{1}{2}}$$

$$= \frac{5}{2}\left(2^2\right)^{\frac{3}{2}} + \frac{1}{2}\left(2^2\right)^{-\frac{1}{2}}$$

$$= \frac{5}{2} \times 2^3 + \frac{1}{2} \times 2^{-1}$$

$$\mathbf{= 20\frac{1}{4}} \checkmark$$

$$\mathbf{= \frac{81}{4}}$$

Question 11 (2 marks)

(✓= 1 mark)

First find $f'(x)$ *and* $g'(x)$ and substitute in $f'(x)g(x) = f(x)g'(x)$.

$f(x) = a + bx^2$

$f'(x) = 2bx$

$g(x) = c + dx^2$

$g'(x) = 2dx$

$\boldsymbol{f'(x)g(x) = f(x)g'(x)}$

$\boldsymbol{2bx \times (c + dx^2) = (a + bx^2) \times 2dx}$✓

$\boldsymbol{2bcx + 2bdx^3 = 2adx + 2bdx^3}$

$\boldsymbol{2bcx = 2adx}$

$\boldsymbol{bc = ad}$✓

Question 12 (3 marks)

(✓= 1 mark)

$y = (x - a)^2 (x - b)$

Identify u and v.

Let $u = (x - a)^2$ and $v = x - b$.

$u' = 2(x - a)$ and $v' = 1$

Write down the expression for $uv' + vu'$.

$$\frac{dy}{dx} = (x - a)^2 \times 1 + (x - b) \times 2(x - a)$$

Expand and simplify.

$$\frac{dy}{dx} = (x - a)((x - a) + 2(x - b))$$

$$= (x - a)(3x - a - 2b) \quad \ldots[1] \checkmark$$

$$\frac{dy}{dx} = (x - 5)(3x - 11) \quad \ldots[2] \checkmark$$

On comparing [1] and [2], we get

$a = 5$ and

$a + 2b = 11$

$2b = 11 - a$

$2b = 11 - 5$

$b = 3$

Therefore, the values of a and b are **5** and **3**.✓

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Question 13 (3 marks)

(✓= 1 mark)

$f(x) = (ax - 4)(ax + 3)$

Use the product rule to find $f'(x)$

Identify u and v.

Let $u = ax - 4$ and $v = ax + 3$.

$u' = a$ and $v' = a$

Write down the expression for $uv' + vu'$.

$$\begin{aligned} f'(x) &= (ax - 4)a + (ax + 3)a \\ &= (ax - 4 + ax + 3)a \\ &= (2ax - 1)a \\ &= 2a^2x - a \ \checkmark \end{aligned}$$

Given that $f'(6) = 6$, substitute the value for x and simplify.

$2a^2(6) - 2a = 6$

$12a^2 - 2a - 6 = 0$ ✓

$6a^2 - a - 3 = 0$

$$a = \frac{-1 \pm \sqrt{(-1)^2 - 4(6)(-3)}}{2(6)} = \frac{-1 \pm \sqrt{73}}{12} \ \checkmark$$

Question 14 (4 marks)

(✓= 1 mark)

a $f(x) = (ax + b)(bx + a)$

Identify u and v.

Let $u = ax + b$ and $v = bx + a$.

$u' = a$ and $v' = b$

Write down the expression for $uv' + vu'$.

$f'(x) = a(bx + a) + b(ax + b)$✓

$= abx + a^2 + abx + b^2$

$= \mathbf{a^2 + b^2 + 2abx}$✓

b Substitute the value for x in the obtained equation.

$f'(1) = a^2 + b^2 + 2ab(1)$

$25 = a^2 + b^2 + 2ab$

$25 = (a + b)^2$

$a + b = 5 \quad \ldots[1]$

$f'(2) = a^2 + b^2 + 2ab(2)$

$37 = a^2 + b^2 + 4ab \ \ldots[2]$

Solving both the equations, we get $\boldsymbol{a = 3, b = 2}$✓ and $\boldsymbol{a = 2, b = 3}$.✓

Question 15 (2 marks)

(✓= 1 mark)

$y = (x - 6)(x^2 - 9)$

$\frac{dy}{dx} = 3x^2 - 12x - 9$ ✓

$3x^2 - 12x - 9 = 0$

$x^2 - 4x - 3 = 0$

$x = \frac{4 \pm \sqrt{28}}{2} = 2 \pm \sqrt{7}$

$x = 2 + \sqrt{7}, y = \left((2 + \sqrt{7}) - 6\right)\left((2 + \sqrt{7})^2 - 9\right) = (20 - 14\sqrt{7})$

$x = 2 - \sqrt{7}, y = \left((2 - \sqrt{7}) - 6\right)\left((2 - \sqrt{7})^2 - 9\right) = (20 + 14\sqrt{7})$

$\mathbf{(2 - \sqrt{7}, 20 + 14\sqrt{7}), (2 + \sqrt{7}, 20 - 14\sqrt{7})}$ ✓

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Question 16 (1 mark)

(✓= 1 mark)

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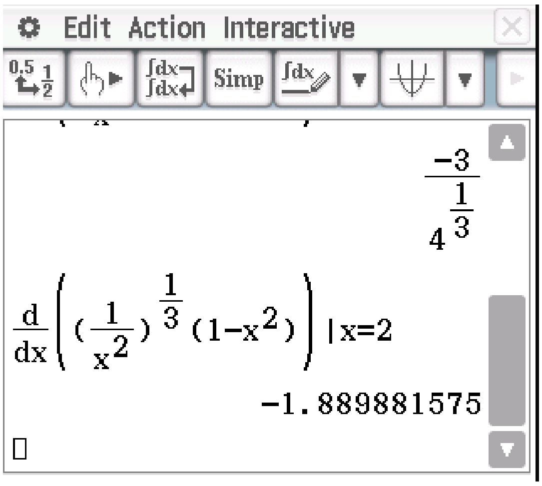

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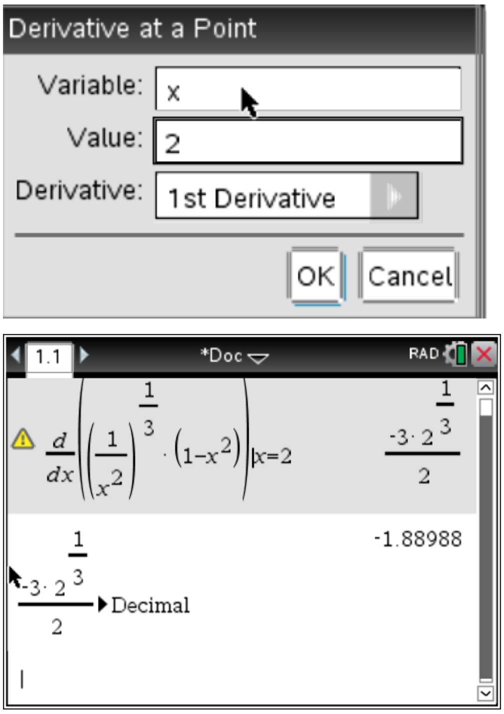

***x* = –1.9**

Question 17 (2 marks)

(✓= 1 mark)

$$\frac{dy}{dx} = \frac{9}{2}x^{\frac{1}{2}} - \frac{5}{2}x^{\frac{3}{2}} \checkmark$$

$$\frac{dy}{dx} = 0, \quad \frac{9}{2}x^{\frac{1}{2}} - \frac{5}{2}x^{\frac{3}{2}} = 0$$

$$x^{\frac{1}{2}}(9-5x) = 0$$

$$x = 0, 9 - 5x = 0$$

$x = \frac{9}{5}$ or 1.8 ($x = 0$ is not a solution because there is no derivative at that point.)

$$y = \left(3 - \frac{9}{5}\right)\sqrt{\left(\frac{9}{5}\right)^3} = \frac{162\sqrt{5}}{125}$$

$$\left(\frac{9}{5}, \frac{162\sqrt{5}}{125}\right) \checkmark$$

Question 18 (2 marks)

(✓= 1 mark)

$$\frac{d}{dx}[x^2(2 - ax^3)] = 4x - 5ax^4 \checkmark$$

$$4x - 5ax^4 = 4x - \frac{5x^4}{2}$$

$$-5ax^4 = -\frac{5x^4}{2}$$

$$a = \frac{1}{2} \checkmark$$

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EXERCISE 1.3 The quotient rule

Question 1

$f(x) = x(\sqrt{x} - 1)$

Identify u and v.

Let $u = x$ and $v = \sqrt{x} - 1$.

$u' = 1$ and $v' = \dfrac{1}{2\sqrt{x}}$

Write down the expression for $uv' + vu'$.

$$f'(x) = x \times \frac{1}{2\sqrt{x}} + \left(\sqrt{x} - 1\right)$$

Substitute the value and simplify.

$$\begin{aligned} f'(4a^2) &= 4a^2 \times \frac{1}{2\sqrt{4a^2}} + 2a - 1 \\ &= 4a^2 \times \frac{1}{4a} + 2a - 1 \\ &= a + 2a - 1 \\ &= 3a - 1 \end{aligned}$$

Therefore, the correct response is **E**.

Question 2

$u(x) = x + 3 \quad v(x) = 2x - 5$

$u'(x) = 1 \qquad v'(x) = 2$

$f'(x) = u(x)v'(x) + u'(x)v(x)$

$= (x + 3) \times 2 + (2x - 5) \times 1$

$= 4x + 1$

$\mathbf{f'(2) = 4 \times 2 + 1 = 9}$

Question 3

Identify u and v.

$u = 6x - 1$ and $v = 9x - 8$

Differentiate to obtain u' and v'.

$u' = 6$ and $v' = 9$

Write down the expression for $\frac{vu' - uv'}{v^2}$.

$$\frac{d}{dx}\left(\frac{6x-1}{9x-8}\right) = \frac{(9x-8)\times 6-(6x-1)\times 9}{(9x-8)^2}$$

$$= \frac{54x-48-54x+9}{(9x-8)^2}$$

$$= \mathbf{\frac{-39}{(9x-8)^2}}$$

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Question 4

a Identify u and v.

$u = 2$ and $v = 2x + 3$

Differentiate to obtain u' and v'.

$u' = 0$ and $v' = 2$

Write down the expression for $\frac{vu'-uv'}{v^2}$.

$$\frac{d}{dx}\left(\frac{2}{2x+3}\right)=\frac{0-2\times 2}{(2x+3)^2}$$
$$=\mathbf{\frac{-4}{(2x+3)^2}}$$

b Identify u and v.

$u = 4 - x$ and $v = x - 5$

Differentiate to obtain u' and v'.

$u' = -1$ and $v' = 1$

Write down the expression for $\frac{vu'-uv'}{v^2}$.

$$\frac{d}{dx}\left(\frac{4-x}{x-5}\right)=\frac{(x-5)\times(-1)-(4-x)\times 1}{(x-5)^2}$$
$$=\frac{-x+5-4+x}{(x-5)^2}$$
$$=\mathbf{\frac{1}{(x-5)^2}}$$

c Identify u and v.

$u = x - 1$ and $v = x + 1$

Differentiate to obtain u' and v'.

$u' = 1$ and $v' = 1$

Write down the expression for $\frac{vu'-uv'}{v^2}$.

$$\frac{d}{dx}\left(\frac{x-1}{x+1}\right)=\frac{(x+1)-(x-1)}{(x+1)^2}$$
$$=\mathbf{\frac{2}{(x+1)^2}}$$

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d Identify u and v.

$u = 1$ and $v = x(x + 1) = x^2 + x$

Differentiate to obtain u' and v'.

$u' = 0$ and $v' = 2x + 1$

Write down the expression for $\frac{vu' - uv'}{v^2}$.

$$\frac{d}{dx}\left(\frac{1}{x(x+1)}\right) = \frac{0 - 1(2x+1)}{(x^2+x)^2}$$

$$= \frac{-(2x+1)}{(x(x+1))^2}$$

$$= \mathbf{\frac{-(2x+1)}{x^2(x+1)^2}}$$

e Identify u and v.

$u = x^3 + x$ and $v = x + 3$

Differentiate to obtain u' and v'.

$u' = 3x^2 + 1$ and $v' = 1$

Write down the expression for $\frac{vu' - uv'}{v^2}$.

$$\frac{d}{dx}\left(\frac{x^3+x}{x+3}\right) = \frac{(x+3)(3x^2+1) - (x^3+x)}{(x+3)^2}$$

$$= \frac{3x^3 + x + 9x^2 + 3 - x^3 - x}{(x+3)^2}$$

$$= \mathbf{\frac{2x^3 + 9x^2 + 3}{(x+3)^2}}$$

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f Identify u and v.

$u = 1 + x + x^2$ and $v = x$

Differentiate to obtain u' and v'.

$u' = 1 + 2x$ and $v' = 1$

Write down the expression for $\frac{vu'-uv'}{v^2}$.

$$\frac{d}{dx}\left(\frac{1+x+x^2}{x}\right) = \frac{x(1+2x)-(1+x+x^2)}{x^2}$$

$$= \frac{x+2x^2-1-x-x^2}{x^2}$$

$$= \boldsymbol{\frac{x^2-1}{x^2}}$$

g $$\left(\frac{x^3-1}{x-1}\right) = \frac{(x-1)(x^2+x+1)}{(x-1)}$$

$$= x^2+x+1$$

$$\frac{d}{dx}\left(\frac{x^3-1}{x-1}\right) = \frac{d}{dx}(x^2+x+1)$$

$$= \mathbf{2x+1}$$

h Identify u and v.

$u = 3(2x + 5) = 6x + 15$ and $v = 1 - x^2$

Differentiate to obtain u' and v'.

$u' = 6$ and $v' = -2x$

Write down the expression for $\frac{vu'-uv'}{v^2}$.

$$\frac{d}{dx}\left(\frac{6x+15}{1-x^2}\right) = \frac{(1-x^2)\times 6-(6x+15)\times(-2x)}{(1-x^2)^2}$$

$$= \frac{6-6x^2+12x^2+30x}{(1-x^2)^2}$$

$$= \boldsymbol{\frac{6x^2+30x+6}{(x^2-1)^2}}$$

Question 5

$u(x) = 2x - 1 \qquad v(x) = x + 4$

$u'(x) = 2 \qquad v'(x) = 1$

$$\frac{vu' - uv'}{v^2}$$

$$= \frac{(x+4)\times 2 - (2x-1)\times 1}{(x+4)^2}$$

$$= \frac{9}{(x+4)^2}$$

At $x = 2$,

$$\frac{9}{(x+4)^2} = \frac{9}{36} = \mathbf{\frac{1}{4}}$$

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Question 6 (4 marks)

(✓ = 1 mark)

Identify u and v.

$u = x + k$ and $v = x - k$

Differentiate to obtain u' and v'.

$u' = 1$ and $v' = 1$

Write down the expression for $\frac{vu'-uv'}{v^2}$.

$$\frac{d}{dx}\left(\frac{x+k}{x-k}\right)=\frac{x-k-x-k}{(x-k)^2}$$
$$=\frac{-2k}{(x-k)^2}$$

$f'(5) = -8$

$$\frac{-2k}{(5-k)^2}=-8$$

$$\frac{2k}{(5-k)^2}=8$$

$2k = (25 + k^2 - 10k) \times 8$

$2k = 200 + 8k^2 - 80k$

$8k^2 - 82k + 200 = 0$

$4k^2 - 41k + 100 = 0$

$(4k - 25)(k - 4) = 0$

Since k is an integer, $\boldsymbol{k = 4}$.

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Question 7 (1 mark)

(✓ = 1 mark)

$$f'(x) = \frac{d}{dx}\left(\frac{1}{2x-4}+3\right)$$

$u(x) = 1 \qquad v(x) = 2x - 4$

$u'(x) = 0 \qquad v'(x) = 2$

$$\frac{vu' - uv'}{v^2}$$

$$= \frac{(2x-4)\times 0 - 1\times 2}{(2x-4)^2}$$

$$f'(x) = \frac{-2}{(2x-4)^2}$$

$$= \frac{-2}{(2)^2(x-2)^2} \checkmark$$

$$= \frac{-1}{2(x-2)^2}$$

Question 8 (2 marks)

(✓= 1 mark)

$u(x) = 2 - x + x^2 \qquad v(x) = x^2 - 2x + 1$

$u'(x) = 2x - 1 \qquad v'(x) = 2x - 2$ ✓

$$\frac{vu' - uv'}{v^2} = \frac{(x^2-2x+1)(2x-1)-(2-x+x^2)(2x-2)}{(x^2-2x+1)^2}$$

For $x = 2$,

$$\frac{vu' - uv'}{v^2} = \frac{(1)(3)-(4)(2)}{(1)^2}$$

$$= \mathbf{-5} \checkmark$$

Question 9 (5 marks)

(✓= 1 mark)

$u(x) = 2 + x \quad v(x) = 2 - x$

$\mathbf{u'(x) = 1} \quad \mathbf{v'(x) = -1}$✓

$$\frac{vu' - uv'}{v^2} = \frac{(2-x)\times 1 - (2+x)\times(-1)}{(2-x)^2}$$

For $x = 4$, $y = -3$.

$$\frac{vu' - uv'}{v^2} = \frac{-2+6}{(-2)^2} = \mathbf{1}$$✓ (gradient at $x = 4$)

Equation of line:

Use point (4, –3).

$y - (-3) = 1(x - 4)$

$\mathbf{y = x - 7}$✓

$y = 0$, $x = 7$, **x-intercept is (7, 0)**✓

$x = 0$, $y = -7$, **y-intercept is (0, –7)**✓

Question 10 (4 marks)

(✓= 1 mark)

Identify u and v.

$u = 5$ and $v = 2x - 1$

Differentiate to obtain u' and v'.

$u' = 0$ and $v' = 2$

Write down the expression for $\frac{vu' - uv'}{v^2}$.

$$\frac{d}{dx}\left(\frac{5}{2x-1}\right) = \frac{0-10}{(2x-1)^2} \checkmark$$
$$= \frac{-10}{(2x-1)^2}$$

Given that $\left(2x-1\right)f'(x) = -\frac{a}{x+b}$ ✓

$$(2x-1)\frac{-10}{(2x-1)^2} = -\frac{a}{x+b}$$

$$\frac{-10}{2x-1} = -\frac{a}{x+b}$$

$$\frac{-10}{2\left(x-\frac{1}{2}\right)} = -\frac{a}{x+b}$$

$$\frac{-5}{x-\frac{1}{2}} = \frac{-a}{x+b} \checkmark$$

$a = 5$ and $b = -\frac{1}{2}$ ✓

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Question 11 (3 marks)

(✓= 1 mark)

Identify u and v.

$u = 1$ and $v = 1 + x$

Differentiate to obtain u' and v'.

$u' = 0$ and $v' = 1$

Write down the expression for $\frac{vu' - uv'}{v^2}$.

Recognise to use quotient rule✓

$$\frac{d}{dx}\left(\frac{1}{1+x}\right) = \frac{0-1}{(1+x)^2}$$

$$= -\frac{1}{(1+x)^2} \checkmark$$

$$= -y^2 \checkmark$$

Question 12 (1 mark)

(✓= 1 mark)

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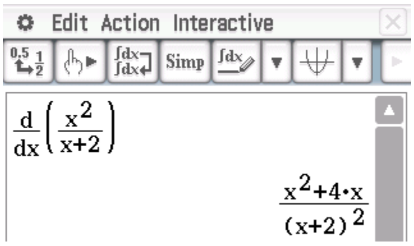

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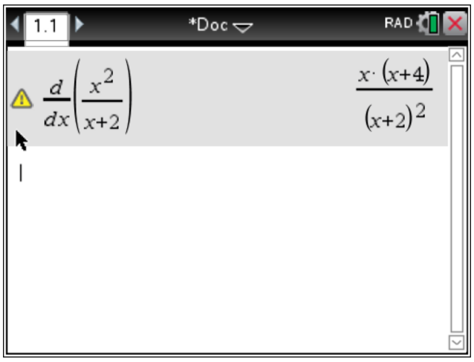

$$\frac{dy}{dx} = -\frac{x^2+4x}{(x+2)^2} \text{ or } \frac{dy}{dx} = -\frac{x(x+4)}{(x+2)^2} \checkmark$$

Question 13 (2 marks)

(✓= 1 mark)

Identify u and v.

$u = x^2$ and $v = x + 1$

Differentiate to obtain u' and v'.

$u' = 2x$ and $v' = 1$

Write down the expression for $\frac{vu' - uv'}{v^2}$.

$$f'(x) = \frac{(x+1)\times 2x - x^2 \times 1}{(x+1)^2}$$
$$= \frac{2x^2 + 2x - x^2}{(x+1)^2}$$
$$= \frac{x^2 + 2x}{(x+1)^2}$$
$$= \frac{x(x+2)}{(x+1)^2}$$

$u = 3x + 2$ and $v = x + 2$

Differentiate to obtain u' and v'.

$u' = 3$ and $v' = 1$

$$g'(x) = \frac{(x+2)\times 3 - (3x+2)\times 1}{(x+2)^2}$$
$$= \frac{3x + 6 - 3x - 2}{(x+2)^2}$$
$$= \frac{4}{(x+2)^2}$$

$f'(x) = g'(x)$

$4(x + 1)^2 = x(x + 2)^3$

$4(x^2 + 2x + 1) = x(x^3 + 6x^2 + 12x + 8)$

$4x^2 + 8x + 4 = x^4 + 6x^3 + 12x^2 + 8x$

$\mathbf{x^4 + 6x^3 + 8x^2 - 4 = 0}$✓

The real values of x that satisfies $f'(x) = g'(x)$ are **–4.11 and 0.58.**✓

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Question 14 (3 marks)

(✓= 1 mark)

$x = 0, y = 1$ y-intercept is (0, 1)

$y = 0,$

$$\frac{x^2 - 4}{x^2 - 3x - 4} = 0$$

$$x^2 - 4 = 0$$

$$x = \pm 2$$

x-intercepts are (–2, 0), (2, 0)✓

Find gradient at the three intercepts using a calculator. (Or use the quotient rule)

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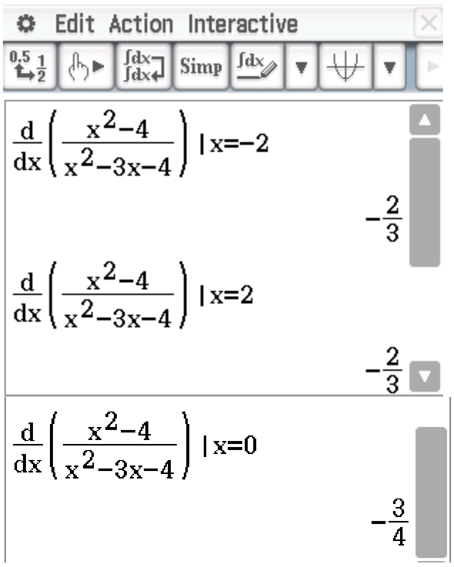

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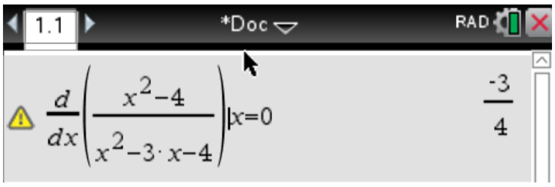

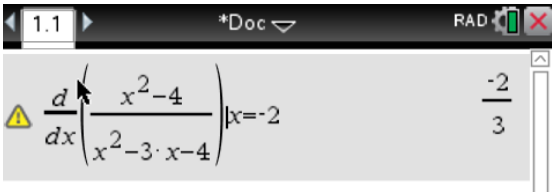

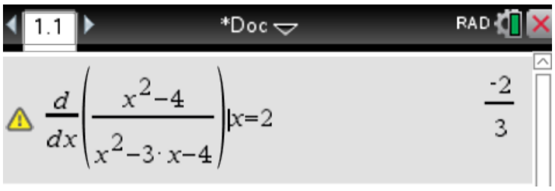

The gradient is $-\frac{3}{4}$ at (0, 1) and $-\frac{2}{3}$ at (–2, 0) and (2, 0)✓

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Question 15 (3 marks)

(✓= 1 mark)

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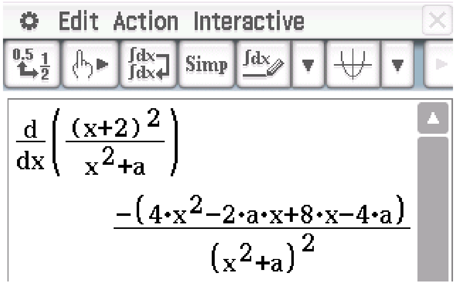

$$\frac{-(4x^2-2ax+8x-4a)}{(x^2+a)^2}=\frac{8-28x-16x^2}{(2x^2+1)^2}$$

$$\frac{-(4x^2-2ax+8x-4a)}{\frac{1}{\sqrt{2}}(2x^2+2a)^2}=\frac{8-28x-16x^2}{(2x^2+1)^2}\ ✓$$

$$2x^2+2a=2x^2+1\Rightarrow \boldsymbol{a}=\frac{\mathbf{1}}{\mathbf{2}}\ ✓$$

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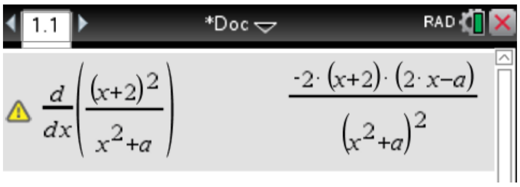

$$\frac{-2(x+2)(2x-a)}{(x^2+a)^2}=\frac{8-28x-16x^2}{(2x^2+1)^2}$$

$$\frac{-2(x+2)(2x-a)}{\frac{1}{\sqrt{2}}(2x^2+2a)^2}=\frac{8-28x-16x^2}{(2x^2+1)^2}\ ✓$$

Question 16 (3 marks)

(✓ = 1 mark)

$$f(x) = ax + \frac{x^2+1}{x+1}$$

Stationary points are found by finding $f'(x) = 0$.

$$\begin{aligned} f'(x) &= a \times 1 + \frac{(x+1)\times 2x - (x^2+1)}{(x+1)^2} \\ &= a + \frac{2x^2 + 2x - x^2 - 1}{(x+1)^2} \\ &= a + \frac{x^2 + 2x - 1}{(x+1)^2} \\ &= \frac{a(x+1)^2 + x^2 + 2x - 1}{(x+1)^2} \quad ✓ \end{aligned}$$

To have at least one stationary point, $f'(x) = 0$.

$a(x + 1)^2 + x^2 + 2x - 1 = 0$

$a(x^2 + 2x + 1) + x^2 + 2x - 1 = 0$

$ax^2 + 2ax + a + x^2 + 2x - 1 = 0$

$(a + 1)x^2 + (2a + 2)x + a - 1 = 0$

To have at least one stationary point, $\Delta \geq 0$

$(2a + 2)^2 - 4(a + 1)(a - 1) \geq 0$ ✓

$4a^2 + 8a + 4 - 4a^2 + 4 \geq 0$

$8a + 8 \geq 0$

$a \geq -1$

Therefore, $\boldsymbol{a \geq -1}$.✓

EXERCISE 1.4 The chain rule

Question 1

$$f(x)=\frac{x^2}{2x-3}$$

Identify u and v.

$u = x^2$ and $v = 2x - 3$

Differentiate to obtain u' and v'.

$u' = 2x$ and $v' = 2$

Write down the expression for $\frac{vu'-uv'}{v^2}$.

$$\begin{aligned}f'(x)&=\frac{(2x-3)\times 2x-x^2\times 2}{(2x-3)^2}\\&=\frac{4x^2-6x-2x^2}{(2x-3)^2}\\&=\frac{2x^2-6x}{(2x-3)^2}\\&=\frac{2x(x-3)}{(2x-3)^2}\end{aligned}$$

Question 2

$$f(x)=\frac{\sqrt[3]{x}}{x+2}$$

Identify u and v.

$u=\sqrt[3]{x}=x^{\frac{1}{3}}$ and $v=x+2$

Differentiate to obtain u' and v'.

$u'=\frac{1}{3}x^{\frac{1}{3}-1}=\frac{1}{3}x^{-\frac{2}{3}}$ and $v'=1$

Write down the expression for $\frac{vu'-uv'}{v^2}$.

$$f'(x)=\frac{(x+2)\times\frac{1}{3}x^{-\frac{2}{3}}-x^{\frac{1}{3}}}{(x+2)^2}$$

$$=-\frac{2(x-1)}{3x^{\frac{2}{3}}(x+2)^2}$$

The gradient of the tangent to the function $f(x)$ at $x=27$ is

$$f'(x)=-\frac{2(27-1)}{3(27)^{\frac{2}{3}}(27+2)^2}$$

$$=-\frac{52}{22\,707}$$

Therefore, the correct response is **A**.

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Question 3

a Write $\frac{2}{\left(x^3+1\right)^4}$ as a function of a function in index form.

Let $u = x^3 + 1$. Then

$$y = \frac{2}{\left(x^3+1\right)^4}$$
$$= 2(x^3 + 1)^{-4}$$
$$= 2u^{-4}$$

Write the chain rule and find the two derivatives.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{du} = -8u^{-5}$$
$$\frac{du}{dx} = 3x^2$$

Substitute the derivatives.

$$\frac{dy}{dx} = -8u^{-5} \times 3x^2$$

Substitute for u.

$$\frac{dy}{dx} = -8(x^3 + 1)^{-5} \times 3x^2$$
$$= -24x^2(x^3 + 1)^{-5}$$
$$\frac{d}{dx}\left(\frac{2}{(x^3+1)^4}\right) = -\mathbf{\frac{24x^2}{\left(x^3+1\right)^5}}$$

b Write $\sqrt{x^2-1}$ as a function of a function.

Let $u = x^2 - 1$. Then

$$y = \sqrt{x^2-1} = \left(x^2-1\right)^{\frac{1}{2}}$$

$$= u^{\frac{1}{2}}$$

Write the chain rule and find the two derivatives.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{du} = \frac{1}{2}u^{\frac{1}{2}-1}$$

$$= \frac{1}{2}u^{-\frac{1}{2}}$$

$$\frac{du}{dx} = 2x$$

Substitute the derivatives.

$$\frac{dy}{dx} = \frac{1}{2}u^{-\frac{1}{2}} \times 2x$$

$$= \frac{x}{u^{\frac{1}{2}}}$$

Substitute for u.

$$\frac{dy}{dx} = \frac{\boldsymbol{x}}{\sqrt{\boldsymbol{x}^2 - \mathbf{1}}}$$

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Question 4

a Let $f(x) = (x - 5)^5$.

Let $u = x - 5$.

Then $y = (x - 5)^5$

$= u^5$

Write the chain rule and find the two derivatives.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{du} = 5u^4$$

$$\frac{du}{dx} = 1$$

Substitute the derivatives.

$$\frac{dy}{dx} = 5u^4$$

Substitute for u.

$$\frac{dy}{dx} = \mathbf{5(x - 5)^4}$$

b Let $f(x) = (4x - 3)^4$.

Let $u = 4x - 3$.

Then $y = (4x - 3)^4$

$= u^4$

Write the chain rule and find the two derivatives.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{du} = 4u^3$$

$$\frac{du}{dx} = 4$$

Substitute the derivatives.

$$\frac{dy}{dx} = 4u^3 \times 4$$

Substitute for u.

$$\frac{dy}{dx} = \mathbf{16(4x - 3)^3}$$

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c Let $f(x) = (2x^3 + x)^3$

Let $u = 2x^3 + x$.

Then $y = (2x^3 + x)^3$

$= u^3$

Write the chain rule and find the two derivatives.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{du} = 3u^2$$

$$\frac{du}{dx} = 6x^2 + 1$$

Substitute the derivatives.

$$\frac{dy}{dx} = 3u^2 \times (6x^2 + 1)$$

Substitute for u.

$$\frac{dy}{dx} = 3(2x^3 + x)^2 (6x^2 + 1)$$

$$= \mathbf{3(6x^2 + 1)(2x^3 + x)^2}$$

d Let $f(x) = (8 - 2x^2)^6$

Let $u = 8 - 2x^2$.

Then $y = (8 - 2x^2)^6$

$= u^6$

Write the chain rule and find the two derivatives.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{du} = 6u^5$$

$$\frac{du}{dx} = -4x$$

Substitute the derivatives.

$$\frac{dy}{dx} = 6u^5 \times (-4x)$$

Substitute for u.

$$\frac{dy}{dx} = 6(8 - 2x^2)^5 \times (-4x)$$

$$= \mathbf{-24x(8 - 2x^2)^5}$$

e

Let $f(x)=\left(\frac{1}{2}x-6\right)^9$.

Let $u=\frac{1}{2}x-6$.

Then, $y=\left(\frac{1}{2}x-6\right)^9$

$=u^9$

Write the chain rule and find the two derivatives.

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

$\frac{dy}{du}=9u^8$

$\frac{du}{dx}=\frac{1}{2}$

Substitute the derivatives.

$\frac{dy}{dx}=9u^8\times\frac{1}{2}$

Substitute for u.

$\frac{dy}{dx}=9\times\left(\frac{1}{2}x-6\right)^8\times\frac{1}{2}$

$=\mathbf{\frac{9}{2}\left(\frac{1}{2}x-6\right)^8}$

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f Let $f(x) = (x^3 - 2x^2 + x + 1)^2$.

Let $u = (x^3 - 2x^2 + x + 1)$.

Then $y = (x^3 - 2x^2 + x + 1)^2$

$= u^2$

Write the chain rule and find the two derivatives.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{du} = 2u$$

$$\frac{du}{dx} = 3x^2 - 4x + 1$$

Substitute the derivatives.

$$\frac{dy}{dx} = 2u\,(3x^2 - 4x + 1)$$

Substitute for u.

$f'(x) = \mathbf{2(3x^2 - 4x + 1)(x^3 - 2x^2 + x + 1)}$

g Let $f(x)=(4x+6)^{\frac{1}{2}}$.

Let $u = 4x + 6$.

Then $y = (4x+6)^{\frac{1}{2}}$

$= u^{\frac{1}{2}}$

Write the chain rule and find the two derivatives.

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

$$\frac{dy}{du}=\frac{1}{2}u^{-\frac{1}{2}}$$

$$\frac{du}{dx}=4$$

Substitute the derivatives.

$$\frac{dy}{dx}=\frac{1}{2}u^{-\frac{1}{2}}\times 4$$

Substitute for u.

$$\begin{aligned} f'(x) &= 2(4x+6)^{-\frac{1}{2}} \\ &= \frac{2}{(4x+6)^{\frac{1}{2}}} \\ &= \frac{\sqrt{2}\times\sqrt{2}}{\sqrt{2}\times\sqrt{2x+3}} \\ &= \boldsymbol{\frac{\sqrt{2}}{\sqrt{2x+3}}} \end{aligned}$$

h Let $f(x) = \left(2\sqrt{x} - x\right)^3$.

Let $u = 2\sqrt{x} - x$.

Then $y = \left(2\sqrt{x} - x\right)^3$

$= u^3$

Write the chain rule and find the two derivatives.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{du} = 3u^2$$

$$\frac{du}{dx} = \frac{2}{2\sqrt{x}} - 1$$

$$= \frac{1}{\sqrt{x}} - 1$$

Substitute the derivatives.

$$\frac{dy}{dx} = 3u^2\left(\frac{1}{\sqrt{x}} - 1\right)$$

Substitute for u.

$$f'(x) = 3\left(2\sqrt{x} - x\right)^2\left(\frac{1}{\sqrt{x}} - 1\right)$$

$$= \frac{3\left(1 - \sqrt{x}\right)\left(x - 2\sqrt{x}\right)^2}{\sqrt{x}}$$

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i Let $f(x) = \sqrt{5(x+10)}$.

Let $u = 5(x+10)$.

Then $y = \sqrt{5(x+10)}$

$= \sqrt{u}$

Write the chain rule and find the two derivatives.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{du} = \frac{1}{2\sqrt{u}}$$

$$\frac{du}{dx} = 5$$

Substitute the derivatives.

$$\frac{dy}{dx} = \frac{1}{2\sqrt{u}} \times 5$$

Substitute for u.

$$f'(x) = \frac{5}{2\sqrt{5(x+10)}}$$

$$= \boldsymbol{\frac{\sqrt{5}}{2\sqrt{x+10}}}$$

j

Let $f(x) = \dfrac{1}{(2x+7)^2}$.

Let $u = 2x+7$.

Then $y = (2x+7)^{-2}$

$= u^{-2}$

Write the chain rule and find the two derivatives.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{du} = -2u^{-3}$$

$$\frac{du}{dx} = 2$$

Substitute the derivatives.

$$\frac{dy}{dx} = -4u^{-3}$$

Substitute for u.

$$f'(x) = \frac{-4}{(2x+7)^3}$$

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k Let $f(x) = \dfrac{1}{\sqrt{4-x}}$.

Let $u = 4 - x$.

Then $y = (4-x)^{-\frac{1}{2}}$

$= u^{-\frac{1}{2}}$

Write the chain rule and find the two derivatives.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{du} = -\frac{1}{2}u^{-\frac{3}{2}}$$

$$\frac{du}{dx} = -1$$

Substitute the derivatives.

$$\frac{dy}{dx} = -\frac{1}{2}u^{-\frac{3}{2}} \times (-1)$$

Substitute for u.

$$f'(x) = -\frac{1}{2}(4-x)^{-\frac{3}{2}} \times (-1)$$

$$= \mathbf{\frac{1}{2\sqrt{(4-x)^3}}}$$

l Let $f(x) = \dfrac{5}{\sqrt{(x-8)^3}}$.

Let $u = x - 8$.

Then $y = 5(x-8)^{-\frac{3}{2}}$

$= 5u^{-\frac{3}{2}}$

Write the chain rule and find the two derivatives.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{du} = -\frac{15}{2}u^{-\frac{5}{2}}$$

$$\frac{du}{dx} = 1$$

Substitute the derivatives.

$$\frac{dy}{dx} = -\frac{15}{2}u^{-\frac{5}{2}}$$

Substitute for u.

$$\boldsymbol{f'(x) = -\frac{15}{2\sqrt{(x-8)^5}}}$$

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Question 5

a Let $f(x) = (2x - 1)^4$.

Let $u = 2x - 1$.

Then $y = (2x - 1)^4$

$= u^4$

Write the chain rule and find the two derivatives.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{du} = 4u^3$$

$$\frac{du}{dx} = 2$$

Substitute the derivatives.

$$\frac{dy}{dx} = 8u^3$$

Substitute for u.

$f'(x) = \mathbf{8(2x - 1)^3}$

b Let $f(x) = (3 - x^3)^2$.

Let $u = 3 - x^3$.

Then $y = (3 - x^3)^2$

$= u^2$

Write the chain rule and find the two derivatives.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{du} = 2u$$

$$\frac{du}{dx} = -3x^2$$

Substitute the derivatives.

$$\frac{dy}{dx} = -6ux^2$$

Substitute for u.

$f'(x) = \mathbf{-6x^2(3 - x^3)}$

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c Let $f(x) = (3 + 4x + 2x^2)^7$.

Let $u = 3 + 4x + 2x^2$.

Then $y = (3 + 4x + 2x^2)^7$

$= u^7$

Write the chain rule and find the two derivatives.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{du} = 7u^6$$

$$\frac{du}{dx} = 4 + 4x$$

Substitute the derivatives.

$$\frac{dy}{dx} = 7u^6 \times (4 + 4x)$$

Substitute for u.

$f'(x) = \mathbf{28(1 + x)(3 + 4x + 2x^2)^6}$

d Let $f(x) = (x^2 + 6x)^6$.

Let $u = x^2 + 6x$.

Then $y = x^2 + 6x$

$= u^6$

Write the chain rule and find the two derivatives.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{du} = 6u^5$$

$$\frac{du}{dx} = 2x + 6$$

Substitute the derivatives.

$$\frac{dy}{dx} = 6u^5 \times (2x + 6)$$

Substitute for u.

$f'(x) = \mathbf{12(x + 3)(x^2 + 6x)^5}$

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e Let $f(x)=\left(x^3-x^6+1\right)^5$.

Let $u = x^3-x^6+1$.

Then $y=(x^3-x^6+1)^5$

$= u^5$

Write the chain rule and find the two derivatives.

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

$$\frac{dy}{du}=5u^4$$

$$\frac{du}{dx}=3x^2-6x^5$$

Substitute the derivatives.

$$\frac{dy}{dx}=5u^4\times(3x^2-6x^5)$$

Substitute for u.

$$f'(x)=15(x^3-x^6+1)^4\left(x^2-2x^5\right)$$

$$= \mathbf{15x^2\left(1-2x^3\right)\left(x^3-x^6+1\right)^4}$$

f Let $f(x) = \frac{1}{n+1}(x^{n+1}+1)^{n+1}$ for positive integers, n

Let $u = x^{n+1}+1$.

Then $y = \frac{1}{n+1}(x^{n+1}+1)^{n+1}$

$= \frac{u^{n+1}}{n+1}$

Write the chain rule and find the two derivatives.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{du} = \frac{(n+1)u^n}{n+1}$$

$$\frac{du}{dx} = (n+1)x^n$$

Substitute the derivatives.

$$\frac{dy}{dx} = \frac{(n+1)u^n}{n+1}(n+1)x^n$$

Substitute for u.

$$f'(x) = \mathbf{(n+1)x^n(x^{n+1}+1)^n}$$

Question 6 (2 marks)

(✓ = 1 mark)

For $y(x) = (3x^2 - 5x)^5$,

Let $y(x) = u^5$ for $u(x) = 3x^2 - 5x$ with $u'(x) = 6x - 5$.

Using the chain rule to differentiate y gives $y'(x) = \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 5u^4 \times u'$.✓

Substituting in the expressions for $u(x)$ and $u'(x)$ gives $\mathbf{y'(x) = 5(6x - 5)(3x^2 - 5x)^4}$.✓

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Question 7 (1 mark)

(✓ = 1 mark)

For $y(x) = (x^2 - 5x)^4$,

let $y = u^4$ for $u(x) = x^2 - 5x$ with $u'(x) = 2x - 5$.

Using the chain rule to differentiate y gives $y'(x) = \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 4u^3 \times u'$.

Substitute in the expressions for $u(x)$ and $u'(x)$.

$\mathbf{y'(x) = 4(x^2 - 5x)^3(2x - 5)}$ **or** $\mathbf{y'(x) = 4x^3(x - 5)^3(2x - 5)}$.✓

Question 8 (1 mark)

(✓ = 1 mark)

Let $y(x) = (4 - x)^{\frac{1}{2}}$ and set $u(x) = 4 - x$ so that $y = u^{\frac{1}{2}}$.

Using the chain rule to differentiate y gives $y'(x) = \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{2}u^{-\frac{1}{2}} \times u'$ where $u'(x) = -1$.

Substituting for $u(x)$ and $u'(x)$ gives $\mathbf{y'(x) = \frac{1}{2}(4 - x)^{-\frac{1}{2}}(-1)}$.

$$\mathbf{y'(x) = -\frac{1}{2\sqrt{4 - x}}} \checkmark$$

Question 9 (1 mark)

(✓ = 1 mark)

$y = \left(-3x^3 + x^2 - 64\right)^3$

Use the chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Let $y = u^3$ $\qquad \frac{dy}{du} = 3u^2$

Let $u = -3x^3 + x^2 - 64$ $\qquad \frac{du}{dx} = -9x^2 + 2x$

$$\mathbf{\frac{dy}{dx} = 3\left(-9x^2 + 2x\right)\left(-3x^3 + x^2 - 64\right)^2} \checkmark$$

Question 10 (2 marks)

(✓ = 1 mark)

$f(x) = (x^2 + ax + 1)^3$ and $f'(0) = 3$

Let $u = x^2 + ax + 1$.

Then $y = (x^2 + ax + 1)^3$

$= u^3$

Write the chain rule and find the two derivatives.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{du} = 3u^2$$

$$\frac{du}{dx} = 2x + a$$

Substitute the derivatives.

$$\frac{dy}{dx} = 3u^2 \times (2x + a)$$

Substitute for u.

$\boldsymbol{f'(x) = 3(x^2 + ax + 1)^2(2x + a)}$✓

$f'(0) = 3(0 + 0 + 1)(0 + a)$

$3 = 3a$

$\boldsymbol{a = 1}$✓

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Question 11 (2 marks)

(✓ = 1 mark)

$y=\sqrt{1+(x-a)^2}$

Let $u = 1+(x-a)^2$.

Then $y=\sqrt{1+(x-a)^2}$

$= u^{\frac{1}{2}}$

Write the chain rule and find the two derivatives.

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

$\frac{dy}{du}=\frac{1}{2}u^{-\frac{1}{2}}$

$\frac{du}{dx}=2(x-a)$

Substitute the derivatives.

$\frac{dy}{dx}=u^{-\frac{1}{2}}(x-a)$

Substitute for u.

$\boldsymbol{\frac{dy}{dx}=\frac{x-a}{\sqrt{1+(x-a)^2}}}$ ✓

$\boldsymbol{\sqrt{1+(x-a)^2}}$ **is positive for all values of x and $x - a$ will be positive if $x > a$.**✓

Question 12 (2 marks)

(✓ = 1 mark)

$y=\sqrt{1-f(x)}=[1-f(x)]^{\frac{1}{2}}$

Let $u(x) = 1 - f(x)$ so that $y=u^{\frac{1}{2}}$.

Following the chain rule, $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$ ✓

Here, $\frac{dy}{du}=\frac{1}{2}u^{-\frac{1}{2}}$ and $\frac{du}{dx}=-f'(x)$.

$$\frac{dy}{dx}=\frac{1}{2}u^{-\frac{1}{2}}\times[-f'(x)].$$

$$=\frac{1}{2}[1-f(x)]^{-\frac{1}{2}}\times[-f'(x)]$$ ✓

$$=\frac{-f'(x)}{2[1-f(x)]^{\frac{1}{2}}}$$

$$=\frac{-f'(x)}{2\sqrt{1-f(x)}}$$

Question 13 (2 marks)

(✓ = 1 mark)

As $f(x) = (x - a)^2g(x)$ is a product of functions, consider $f(x) = u(x)g(x)$, where $u(x) = (x - a)^2$.✓

Differentiating $f(x)$ using the product rule gives

$f'(x) = u'(x)g(x) + u(x)g'(x) = 2(x - a)g(x) + (x - a)^2g'(x)$.

Extracting the common factor from this expression gives
$f'(x) = (x - a)[2g(x) + (x - a)g'(x)]$.✓

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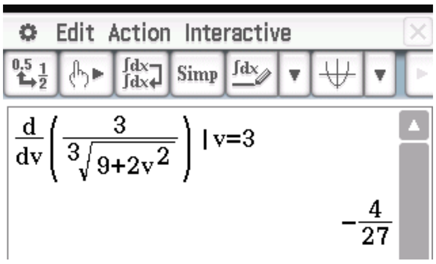
Question 14 (2 marks)

(✓= 1 mark)

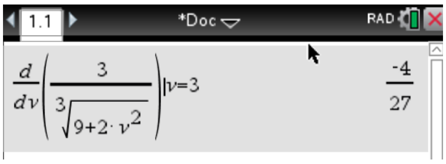
ClassPad



correct method✓

correct answer✓

TI-Nspire



correct method✓

correct answer✓

Question 15 (3 marks)

(✓ = 1 mark)

$f(x) = a(bx+1)^3$, $f(0) = 2$ and $f'(0) = 18$.

$f(0) = a(0+1)^3 = a$

$2 = a$

i.e., $\boldsymbol{a = 2}$✓

Let $u = bx + 1$.

Then $y = a(bx+1)^3$

$= au^3$

Write the chain rule and find the two derivatives.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{du} = 3au^2$$

$$\boldsymbol{\frac{du}{dx} = b}✓$$

Substitute the derivatives.

$$\frac{dy}{dx} = 3abu^2$$

Substitute for u.

$f'(x) = 3ab(bx+1)^2$

Since $f'(0) = 18$

$3ab(0+1)^2 = 18$

$3ab = 18$

$3(2)b = 18$

$6b = 18$

$\boldsymbol{b = 3}$✓

Question 16 (3 marks)

(✓ = 1 mark)

Given that

$f'[g(x)] = \sqrt{a(6x-7)^b}$

$f(x) = \sqrt{x^3}$

$g(x) = 6x - 7$

Differentiating $f(x)$.

$f'(x) = \frac{3}{2}x^{\frac{1}{2}} = \frac{3\sqrt{x}}{2}$

$$\boldsymbol{f'[g(x)] = \frac{3}{2}\sqrt{6x-7}}$$
$$\boldsymbol{= \sqrt{\frac{9}{4}(6x-7)}}\ ✓$$

Comparing to the standard form, we get

$a = \frac{9}{4} = 2.25$ and $b = 1$

Therefore, $\boldsymbol{a = 2.25}$✓ **and b = 1.**✓

Question 17 (3 marks)

(✓ = 1 mark)

$\boldsymbol{h(c) = c^2 + c + 1, c(t) = t^3 + t}$ ✓

Differentiating $h(c)$.

$\frac{dh}{dt} = \frac{dh}{dc} \times \frac{dc}{dt}$

$\boldsymbol{\frac{dh}{dt} = (2t^3 + 2t + 1)(3t^2 + 1)}$✓

When $t = 10$ cm $= 0.1$ m, $\frac{dh}{dt} = (2(0.1)^3 + 2(0.1) + 1)(3(0.1)^2 + 1) =$ **1.2 cm/m**✓

EXERCISE 1.5 Combining the rules

Question 1

$f(x) = (1 - 0.5x^4)^4$ and $x = 1$

Write u as a function of a function.

Let $u(x) = p(q(x))$, where $p(q) = q^4$ and $q(x) = 1 - 0.5x^4$.

Differentiating u using chain rule.

$$u' = \frac{dp}{dx} = \frac{dp}{dq} \times \frac{dq}{dx}$$

Substitute the derivatives.

$$u' = 4q^3(-2x^3)$$

Substitute for q.

$u' = -4(1 - 0.5x^4)^3 2x^3$

$f'(x) = -4(1 - 0.5x^4)^3 2x^3$

Substituting $x = 1$ in $f'(x)$, we get

$$\begin{aligned} f'(1) &= -4(1 - 0.5)^3 \times 2 \\ &= -8(0.5)^3 \\ &= -1 \end{aligned}$$

Therefore, the correct response is **A**.

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Question 2

$f(x) = \sqrt{(x^2+1)^3} = (x^2+1)^{\frac{3}{2}}$

Let $u = x^2 + 1$.

Then $y = (x^2+1)^{\frac{3}{2}}$

$= u^{\frac{3}{2}}$

Write the chain rule and find the two derivatives.

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{du} = \frac{3}{2}u^{\frac{1}{2}}$

$\frac{du}{dx} = 2x$

Substitute the derivatives.

$\frac{dy}{dx} = (2x) \times \frac{3}{2}u^{\frac{1}{2}}$

Substitute for u.

$f'(x) = 3x(x^2+1)^{\frac{1}{2}}$

$= 3x\sqrt{x^2+1}$

Therefore, the correct response is **E**.

Question 3

$f(x) = x^2(2x + 1)^3$

Let $y = uv$, where $u = x^2$ and $v = (2x + 1)^3$.

Find the derivative of u.

$u' = 2x$

Write v as a function of a function.

Let $v(x) = p(q(x))$, where $p(q) = q^3$ and $q(x) = 2x + 1$.

Differentiating v using chain rule.

$$v' = \frac{dp}{dx} = \frac{dp}{dq} \times \frac{dq}{dx}$$

Substitute the derivatives.

$v' = 3q^2 \times 2$

Substitute for q.

$v' = 6(2x + 1)^2$

Write the product rule

$y = uv' + vu'$

Substitute the functions and simplify.

$y = x^2 \times 6(2x + 1)^2 + (2x + 1)^3 \times 2x$

$\quad = \mathbf{2x(5x + 1)(2x + 1)^2}$

Question 4

For $f(x) = (x - 1)^2(x - 2) + 1$,

let $u(x) = x - 1$ so that $f(x) = u^2(u - 1) + 1 = u^3 - u^2 + 1$.

Using the chain rule to differentiate f gives

$$f'(x) = \frac{df}{du} \times \frac{du}{dx} = (3u^2 - 2u) \times 1 = u(3u - 2)$$

Substituting $u(x) = x - 1$ gives $f'(x) = (x - 1)[3(x - 1) - 2] = (x - 1)(3x - 5)$.

Comparing this expression with $f'(x) = (x - 1)(ux + v)$ shows that $\boldsymbol{u = 3}$ and $\boldsymbol{v = -5}$.

Question 5

Let $f(p) = 10p(1 - p)^9$ and use the product rule with $u = 10p$, $v = (1 - p)^9$.

Then $u' = 10$, $v' = -9(1 - p)^8$.

$f'(p) = 10(1 - p)^9 - 90p(1 - p)^8 = 10(1 - p)^8[(1 - p) - 9p)] = \mathbf{10(1 - p)^8(1 - 10p)}$

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Question 6 (6 marks)

(✓ = 1 mark)

a **i** $f(x)=\frac{x+1}{2x-1}$

Identify u and v.

Here, $\boldsymbol{u = x + 1}$ **and** $\boldsymbol{v = 2x - 1}$✓

Differentiating u and v.

$u' = 1$ and $v' = 2$

$$f'(x)=\frac{u'v-v'u}{v^2}$$
$$=\frac{(2x-1)\times 1-2\times(x+1)}{(2x-1)^2}$$
$$=\frac{-3}{(2x-1)^2}\ ✓$$

ii $f(x)=(x+1)(2x-1)^{-1}$

Let $u = x + 1$ and $v = (2x - 1)^{-1}$

$\frac{du}{dx}=1$ and $\frac{dv}{dx}=-2(2x-1)^{-2}$, (chain rule)

So,

$$f'(x)=u'v+uv'$$
$$=(2x-1)^{-1}\times 1+(x+1)\left(-2(2x-1)^{-2}\right)\ ✓$$
$$=(2x-1)^{-1}-2(x+1)(2x-1)^{-2}$$
$$=\frac{1}{2x-1}-\frac{2(x+1)}{(2x-1)^2}\ ✓$$

b

$$\frac{1}{2x-1}-\frac{2(x+1)}{(2x-1)^2}=\frac{1\times(2x-1)}{(2x-1)^2}-\frac{2(x+1)}{(2x-1)^2}\ ✓$$
$$=\frac{2x-1-2x-2}{(2x-1)^2}$$
$$=\frac{-3}{(2x-1)^2}\ ✓$$

Question 7 (4 marks)

(✓ = 1 mark)

Given,

$$f(x)=\frac{x+a}{x-a}$$

$$\boldsymbol{f'(x)=\frac{(x-a)-(x+a)}{(x-a)^2}}\ ✓$$

$$\boldsymbol{=\frac{-2a}{(x-a)^2}}$$

At $x = 1$, the gradient of $f(x)$ is $-\frac{3}{2}$.

Therefore,

$$\boldsymbol{\frac{-2a}{(1-a)^2}=-\frac{3}{2}}\ ✓$$

$$-4a=-3(1-a)^2$$

$$4a=3+3a^2-6a$$

$$\boldsymbol{3a^2-10a+3=0}\ ✓$$

Solving, we get the valid value of ***a* is 3**.✓

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Question 8 (4 marks)

(✓ = 1 mark)

$$y=\frac{x}{\sqrt{a^2-x^2}}$$

$\boldsymbol{u=x,\ v=\sqrt{a^2-x^2}}$ ✓

Differentiating u and v.

$\boldsymbol{u'=1 \text{ and } v'=-x\left(a^2-x^2\right)^{-\frac{1}{2}}}$ ✓

$$\frac{dy}{dx}=\frac{u'v-v'u}{v^2}$$

$$=\frac{\sqrt{a^2-x^2}\times 1+x\left(a^2-x^2\right)^{-\frac{1}{2}}\times x}{\left(\sqrt{a^2-x^2}\right)^2}$$ ✓

$$=\frac{\sqrt{a^2-x^2}+\frac{x^2}{\sqrt{a^2-x^2}}}{a^2-x^2}$$

$$=\frac{\frac{a^2-x^2+x^2}{\sqrt{a^2-x^2}}}{a^2-x^2}$$

$$=\frac{\frac{a^2}{\sqrt{a^2-x^2}}}{a^2-x^2}$$

$$=\frac{a^2}{\left(a^2-x^2\right)^{\frac{1}{2}}\left(a^2-x^2\right)}$$

$$=\frac{a^2}{\left(a^2-x^2\right)^{\frac{3}{2}}}$$ ✓

$$=\frac{a^2}{\sqrt{\left(a^2-x^2\right)^3}}$$

Question 9 (3 marks)

(✓ = 1 mark)

$y = \sqrt{x^2 + 3}$

$x(t) = 4t^3 + t + 1$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$= \frac{x}{\sqrt{x^2 + 3}} \times \left(12t^2 + 1\right)$$ ✓

Substituting $x = 4t^3 + t + 1$

$$\frac{dy}{dt} = \frac{4t^3 + t + 1}{\sqrt{(4t^3 + t + 1)^2 + 3}} \times \left(12t^2 + 1\right)$$ ✓

At $t = 0$, $\frac{dy}{dt} = \frac{1}{2} = 0.5$. ✓

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Question 10 (4 marks)

(✓ = 1 mark)

$$f(x)=\frac{ax^2+b}{x+b}$$

Identify u and v.

Here, $u = ax^2 + b$ and $v = x + b$.

Differentiating u and v.

$u' = 2ax$ and $v' = 1$

$$\begin{aligned}\boldsymbol{f'(x)} &= \frac{u'v-v'u}{v^2}\\ &= \frac{(x+b)\times 2ax-(ax^2+b)\times 1}{(x+b)^2} \quad ✓\\ &= \frac{2ax^2+2abx-ax^2-b}{(x+b)^2}\\ &= \frac{ax^2+2abx-b}{(x+b)^2}\end{aligned}$$

Given. $f(1) = 1$ and $f'(-1)=-2$

$$f(1)=\frac{a+b}{1+b}$$

$$\frac{a+b}{1+b}=1$$

$$a+b=1+b$$

$\boldsymbol{a=1}$ ✓

$$\begin{aligned}f'(-1) &= \frac{a-2ab-b}{(-1+b)^2}\\ -2 &= \frac{1-2b-b}{(-1+b)^2}\\ -2 &= \frac{1-3b}{b^2+1-2b} \quad ✓\end{aligned}$$

$$-2b^2-2+4b=1-3b$$

$$-2b^2+7b-3=0$$

Solving the above equation, we get $b = 0.5$ and 3.

As b is an integer, $\boldsymbol{b = 3}$.✓

Question 11 (3 marks)

(✓ = 1 mark)

Given,

$f(x) = mx$

$\boldsymbol{f'(x) = m}$ ✓

$g(x) = \left(f(x)\right)^n$

Differentiate $g(x)$,

$$g'(x) = n\left(f(x)\right)^{n-1} f'(x)$$

$$= \frac{\boldsymbol{n\left(f(x)\right)^n m}}{\boldsymbol{f(x)}} \checkmark$$

$$= \frac{nmg(x)}{mx}$$

$$= \frac{\boldsymbol{ng(x)}}{\boldsymbol{x}} \checkmark$$

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Question 12 (4 marks)

(✓ = 1 mark)

a $f(x) = (x^2 - x + 1)^2$

Let $u = x^2 - x + 1$.

Then $y = \left(x^2 - x + 1\right)^2$

$= u^2$

Write the chain rule and find the two derivatives.

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{du} = 2u$

$\frac{du}{dx} = 2x - 1$

Substitute the derivatives.

$\frac{dy}{dx} = 2u(2x - 1)$

Substitute for u.

$\boldsymbol{f'(x) = 2(2x - 1)(x^2 - x + 1)}$✓

$g(x) = (x + a)^3$

Let $u = x + a$.

Then $g(x) = (x + a)^3$

$= u^3$

Write the chain rule and find the two derivatives.

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{du} = 3u^2$

$\frac{du}{dx} = 1$

Substitute the derivatives.

$\frac{dy}{dx} = 3u^2$

Substitute for u.

$\boldsymbol{g'(x) = 3(x + a)^2}$✓

b $f'(x) = 2(2x - 1)(x^2 - x + 1)$

$f'(0) = -2$

$g'(x) = 3(x + a)^2$

$g'(0) = 3a^2$

$\boldsymbol{f'(0) \times g'(0) = -1}$ ✓

$-2 \times 3a^2 = -1$

$\boldsymbol{6a^2 = 1}$ ✓

Question 13 (3 marks)

(✓ = 1 mark)

Use the quotient rule to differentiate $\frac{x^2}{1+x^2}$

$u = x^2$ and $v = 1 + x^2$

Differentiating u and v.

$u' = 2x$ and $v' = 2x$

$$\frac{u'v - v'u}{v^2} = \frac{(1+x^2)(2x) - (x^2)(2x)}{(1+x^2)^2} \checkmark$$

$$\frac{dy}{dx} = \frac{2x}{(1+x^2)^2} \checkmark$$

$$(1+x^2)\frac{dy}{dx} + 2xy$$

$$= (1+x^2)\frac{2x}{(1+x^2)^2} + 2x\left(\frac{x^2}{1+x^2}\right)$$

$$= \frac{2x}{(1+x^2)} + \frac{2x^3}{(1+x^2)} \checkmark$$

$$= \frac{2x+2x^3}{(1+x^2)}$$

$$= \frac{2x(1+x^2)}{(1+x^2)}$$

$$= 2x$$

As required.

Question 14 (4 marks)

(✓ = 1 mark)

a Given,

$$f(x) = \frac{\left(6 - x^2\right)x}{3\left(2 - x^2\right)}$$

Differentiate $f(x)$.

$$f'(x) = \frac{3\left(2 - x^2\right)\left(-2x^2 + 6 - x^2\right) - \left(6 - x^2\right)x\left(-6x\right)}{9\left(2 - x^2\right)^2}$$

$$= \frac{-6x^2 + 3x^4 + 12 - 6x^2 + 12x^2 - 2x^4}{3\left(2 - x^2\right)^2}$$

$$= \frac{x^4 + 12}{3\left(x^2 - 2\right)^2}$$

Comparing to the given form, ***a* = 1**✓ and ***b* = 12**.✓

b The coordinates of the point on the curve of $f(x)$ in [0, 5] where $f(x) = f'(x)$ is

$$\frac{x^4 + 12}{3\left(x^2 - 2\right)^2} = \frac{\left(6 - x^2\right)x}{3\left(2 - x^2\right)}$$

Solving the above equation will provide the real value of $x =$ **3.231**.✓

Therefore, the y coordinate is

$$f(3.231) = \frac{(6 - 3.231^2)3.231}{3(2 - 3.231^2)}$$

$$= \mathbf{0.567}$$

The coordinates are **(3.231, 0.567)**.✓

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Question 15 (4 marks)

(✓ = 1 mark)

$u = a + 4x^2 \Rightarrow u' = 8x$ ✓

$v = 2x + 1 \Rightarrow v' = 2$

$$\frac{vu' - uv'}{v^2} = \frac{(2x+1)\times 8x - (a + 4x^2)\times 2}{(2x+1)^2}$$ ✓

$$y' = \frac{8x^2 + 8x - 2a}{(2x+1)^2}$$

$y'(2) = \frac{6}{5}$, so

$$\frac{8(2)^2 + 8(2) - 2a}{(5)^2} = \frac{6}{5}$$

$48 - 2a = 30 \Rightarrow a = 9$ ✓

$y(2) = b,$

$$b = \frac{a + 16}{5}$$

$$b = \frac{9 + 16}{5}$$

$$b = \frac{25}{5}$$

$b = 5$ ✓

or

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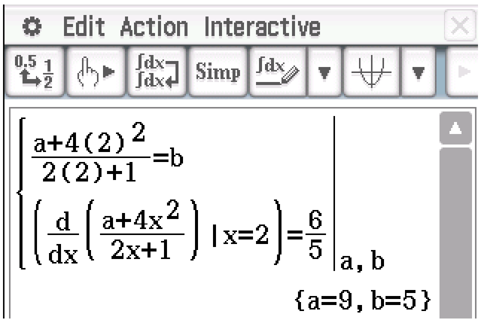

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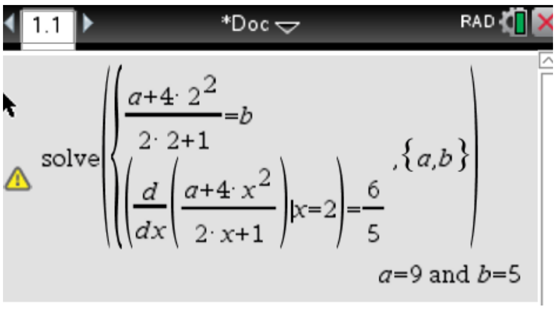

Cumulative examination: Calculator-free

Question 1 (1 mark)

(✓ = 1 mark)

$y=(5x+1)^7$

Use the chain rule $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

Let $u=5x+1$ $\quad\frac{du}{dx}=5$

Let $y=u^7$ $\quad\frac{dy}{du}=7u^6$

$\frac{dy}{dx}=7(5x+1)^6\times 5=\mathbf{35(5x+1)^6}$ ✓

Question 2 [SCSA MM2018 Q3a] (2 marks)

$$\frac{d}{dx}(2x^3+1)^5=5\times(6x^2)(2x^3+1)^4$$
$$=30x^2(2x^3+1)^4$$

demonstrates use of the chain rule by including the $(2x^3+1)^4$ term✓
fully determines derivative correctly✓

Question 3 (4 marks)

(✓ = 1 mark)

a $f(x)=\dfrac{x}{x+2}$

Use the quotient rule $\dfrac{dy}{dx}=\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$

Let $u=x$ $\quad\dfrac{du}{dx}=1$

Let $v=x+2$ $\quad\dfrac{dv}{dx}=1$

$$f'(x)=\frac{(x+2)\times 1-x\times 1}{(x+2)^2}\checkmark$$

$$=\frac{x+2-x}{(x+2)^2}$$

$$=\mathbf{\frac{2}{(x+2)^2}}\checkmark$$

b $g(x)=(2-x^3)^3$

Use the chain rule $\dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}$

Let $u=2-x^3$ $\quad\dfrac{du}{dx}=-3x^2$

Let $y=u^3$ $\quad\dfrac{dy}{du}=3u^2$

$$g'(x)=(-3x^2)\times 3(2-x^3)^2$$

$$=\mathbf{-9x^2(2-x^3)^2}\checkmark$$

$$g'(1)=-9\times 1^2\times(2-1^3)^2$$

$$=-9\times 1^2$$

$$=\mathbf{-9}\checkmark$$

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Question 4 (1 mark)

(✓ = 1 mark)

$$f(x)=\sqrt{1-2x}=(1-2x)^{\frac{1}{2}}$$

$$f'(x)=\frac{1}{\cancel{2}}\times(1-2x)^{-\frac{1}{2}}\times(-\cancel{2})$$

$$=-\frac{1}{(1-2x)^{\frac{1}{2}}}$$

$$=-\frac{\mathbf{1}}{\sqrt{\mathbf{1-2x}}}$$ ✓

Question 5 (3 marks)

(✓ = 1 mark)

Find equation of tangent at (2, 4).

$y = x^2$.

$\frac{dy}{dx} = 2x$

$f'(2) = 2(2) = 4$. ✓

So gradient of line = m = 4, and it passes through (2, 4).

$y - 4 = m(x - 2)$

$y - 4 = 4(x - 2)$

$y - 4 = 4x - 8$

$y = 4x - 4$. ✓

Substitute in values from (3, 8) to see if it lies on the line.

LHS = y = 8

RHS = $4x - 4 = 4(3) - 4 = 12 - 4 = 8$ = LHS, so (3, 8) lies on the tangent. ✓

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Question 6 (1 mark)

(✓ = 1 mark)

Differentiate $\frac{1}{5}(x-2)^2(5-x)$.

Let $u = (x-2)^2$ and $v = (5-x)$

Find u' using the chain rule.

$u' = 2(x-2)(1) = 2(x-2)$

$v' = -1$

$f'(x) = uv' + vu'$

$= \frac{1}{5}[(x-2)^2(-1) + (5-x)2(x-2)]$

$= \frac{1}{5}(x-2)[-x+2+2(5-x)]$

$= \frac{1}{5}(x-2)[-x+2+10-2x]$

$= \frac{1}{5}(x-2)[-3x+12]$

$= -\frac{3}{5}\mathbf{(x-2)(x-4)}$ ✓

Question 7 [SCSA MM2019 Q2ab] (4 marks)

(✓ = 1 mark)

a

$$\left(\frac{g}{h}\right)'(3) = \frac{g'(3)h(3) - g(3)h'(3)}{\left[h(3)\right]^2}$$
$$= \frac{4(6) - (-3)(-5)}{6^2}$$
$$= \frac{1}{4}$$

expresses the derivative using the quotient rule✓

evaluates the derivative✓

b

$$h(g(1))' = h'(g(1))g'(1)$$
$$= h'(3)(-4)$$
$$= (-5)(-4)$$
$$= 20$$

expresses the derivative using the chain rule✓
expresses the derivative✓

Cumulative examination: Calculator-assumed

Question 1 (2 marks)

(✓ = 1 mark)

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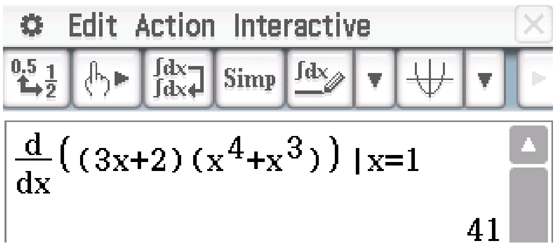

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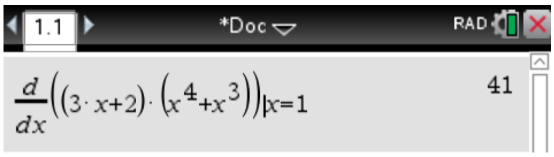

Use a calculator or the product rule to differentiate $f(x)$ and obtain $\mathbf{f'(1)=41}$.✓

$f(1) = (5)(2) = 10$. Use the co-ordinates (1, 10) to find the equation of the tangent.

$y - 10 = 41(x - 1)$

$\mathbf{y = 41x - 31}$ ✓

Question 2 (4 marks)

(✓ = 1 mark)

Use quotient together with the chain rule.✓

$u=(5-x)^3,\quad u'=-3(5-x)^2$

$v=\sqrt{2x+1},\quad v'=\dfrac{1}{\sqrt{2x+1}}$ ✓

$$\frac{dy}{dx}=\frac{\left(\sqrt{2x+1}\right)\left(-3(5-x)^2\right)-(5-x)^3\times\dfrac{1}{\sqrt{2x+1}}}{\left(\sqrt{2x+1}\right)^2}\ ✓$$

$$=\frac{-3(2x+1)(5-x)^2-(5-x)^3}{(2x+1)^{\frac{3}{2}}}$$

$$=\frac{(5-x)^2\left(-3(2x+1)-(5-x)\right)}{(2x+1)^{\frac{3}{2}}}$$

$$=\frac{(5-x)^2(-5x-8)}{(2x+1)^{\frac{3}{2}}}$$

$$=\frac{-(5-x)^2(5x+8)}{(2x+1)^{\frac{3}{2}}}\ ✓$$

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Question 3 (4 marks)

(✓ = 1 mark)

$y(1) = -3$, so $-3 = \dfrac{a+b}{-3}$ or $\boldsymbol{a+b=9}$ ✓

Use quotient rule.

$$\frac{dy}{dx} = \frac{-(2a+5b)}{(2x-5)^2} \checkmark$$

$$x = 1,\quad \frac{dy}{dx} = -\frac{11}{3}$$

$$\frac{-(2a+5b)}{9} = -\frac{11}{3} \text{ or } \boldsymbol{2a+5b=33} \checkmark$$

Solve the simultaneous equations $a + b = 9$ and $2a + 5b = 33$

$\boldsymbol{a = 4, b = 5}$✓

Question 4 (4 marks)

(✓ = 1 mark)

$$\frac{dy}{dx} = x^2 + 4x + 3 \checkmark$$

For stationary points, $\dfrac{dy}{dx} = x^2 + 4x + 3 = 0$ ✓

$$\boldsymbol{(x+1)(x+3) = 0}$$

$$\boldsymbol{x = -3, x = -1} \checkmark$$

$x = -3,\ y = -9 + 18 - 9 - 2 = -2$ coordinates (3, –2)

$$x = -1,\ y = -\frac{1}{3} + 2 - 3 - 2 = -\frac{10}{3}$$

Stationary points at (–3, –2) and $\boldsymbol{\left(-1, -\dfrac{10}{3}\right)}$ ✓

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Chapter 2 – Applications of differentiation

EXERCISE 2.1 Increasing and decreasing functions

Question 1

$$\frac{dy}{dx}=\frac{3x+1+6\sqrt{x}}{2\sqrt{x}\left(\sqrt{x}+1\right)^2}$$

$$\delta x=5.01-5=0.01$$

$$\delta y\approx\frac{3x+1+6\sqrt{x}}{2\sqrt{x}\left(\sqrt{x}+1\right)^2}\delta x$$

$$=\frac{3(5)+1+6\sqrt{5}}{2\sqrt{5}\left(\sqrt{5}+1\right)^2}\times 0.01$$

$$\mathbf{=0.0063}$$

Question 2

$$V=\frac{4}{3}\pi r^3$$

$$\frac{dV}{dr}=4\pi r^2$$

$$\delta r=3.95-4=-0.05$$

$$\delta V\approx 4\pi r^2\delta r$$

$$=4\pi\times 4^2\times -0.05$$

$$\mathbf{=-10.05\ cm^3}$$

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Question 3

$$\frac{dy}{dx} = f'(x) = \frac{1}{2\sqrt{x+1}} - 8x$$

$$\delta x = 3.02 - 3 = 0.02$$

$$\delta y \approx \left(\frac{1}{2\sqrt{x+1}} - 8x\right)\delta x$$

$$= \left(\frac{1}{2\sqrt{3+1}} - 8\times 3\right)\times 0.02$$

$$\mathbf{= -0.475}$$

Question 4 (3 marks)

(✓ = 1 mark)

$$y = \sqrt{x}$$

$$\boldsymbol{\frac{dy}{dx} = \frac{1}{2\sqrt{x}}} \checkmark$$

$$\delta x = 9.01 - 9 = 0.01$$

$$\boldsymbol{\delta y \approx \frac{1}{2\sqrt{x}}\delta x} \checkmark$$

$$= \left(\frac{1}{2\sqrt{9}}\right)\times 0.01$$

$$= \frac{1}{600}$$

Hence $\sqrt{9.01} \approx \sqrt{9} + \frac{1}{600}$

$$\boldsymbol{= 3\frac{1}{600}} \checkmark$$

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Question 5 (4 marks)

(✓ = 1 mark)

$\frac{dy}{dx} = -\frac{8}{x^3} - \frac{1}{2\sqrt{x}}$ ✓

$\delta x = 1.01 - 1 = 0.01$ ✓

$\delta y \approx \left(-\frac{8}{x^3} - \frac{1}{2\sqrt{x}}\right)\delta x$ ✓

$\delta y \approx \left(-\frac{8}{1^3} - \frac{1}{2\sqrt{1}}\right) \times 0.01$

$= -\frac{17}{200}$ ✓

Question 6 (4 marks)

(✓ = 1 mark)

$A = x^2$ Area of square of side length x.

$\frac{dA}{dx} = 2x$ ✓

$\delta x = 0.1$ ✓

$\delta A \approx 2x\delta x$ ✓

$\delta A \approx 2 \times 8 \times 0.1$

$= 1.6\ \text{cm}^2$ ✓

Question 7 (4 marks)

(✓ = 1 mark)

Surface area is $A = 4\pi r^2$

$A = 12.2, \quad 12.2 = 4\pi r^2 \Rightarrow \mathbf{r \approx 0.9853}$ ✓

$\boldsymbol{\delta A = 12.15 - 12.2 = -0.05}$ ✓

$$\frac{dA}{dr} = 8\pi r$$

$$\boldsymbol{\delta A = 8\pi r \delta r}$$

$$\boldsymbol{\delta r = \frac{\delta A}{8\pi r}}$$ ✓

$$\delta r = \frac{-0.05}{8\pi(0.9853)}$$

$$= \mathbf{-0.002\ cm}$$ ✓

Question 8 (4 marks)

(✓ = 1 mark)

Volume is $V = \pi r^2 h$

$\boldsymbol{\frac{dV}{dr} = 2\pi rh}$, **since height *h* is constant.**✓

$\boldsymbol{\delta r = 0.503 - 0.5 = 0.003}$ ✓

$\boldsymbol{\delta V = 2\pi rh\delta r}$ ✓

$$= 2\pi \times 0.5 \times 10 \times 0.003$$

$$= \mathbf{0.094\ cm^3}$$ ✓

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EXERCISE 2.2 Second derivative and points of inflection

Question 1

The perimeter, P, of a rectangle of width x $P = 2x + 3(2x) = 8x$

$$\frac{dP}{dx} = 8$$

$$\delta P = 8\delta x$$

$$\delta x = 2.01 - 2 = 0.01$$

$$\delta P = 8 \times 0.01$$

$$= \mathbf{0.08\ cm}$$

Question 2

Surface area is $A = 4\pi r^2$

$$r = \frac{25}{2} = 12.5$$

Change in diameter 24.9 – 25 = –0.1 so change in radius is −0.05.

$$\delta r = 12.15 - 12.2 = -0.05$$

$$\frac{dA}{dr} = 8\pi r$$

$$\delta A = 8\pi r \delta r$$

$$= 8\pi \times 12.5 \times (-0.05)$$

$$= \mathbf{-15.71\ cm^2}$$

Question 3

a First derivative

$$y' = \frac{1}{2}(x-3)^{-\frac{1}{2}}$$

Second derivative

$$y'' = -\frac{1}{4}(x-3)^{-\frac{3}{2}}$$

$$\mathbf{= -\frac{1}{4(x-3)^{\frac{3}{2}}}}$$

b First derivative using quotient rule

$$f'(x) = \frac{(2x+10)(2x)-(x^2-1)(2)}{(2x+10)^2} = \frac{2x^2+20x+2}{(2x+10)^2}$$

Second derivative using quotient rule

$$f''(x) = \frac{(2x+10)^2(4x+20)-(2x^2+20x+2)(4(2x+10))}{(2x+10)^4}$$

$$= \frac{2(2x+10)^2(2x+10)-4(2x^2+20x+2)(2x+10)}{(2x+10)^4}$$

$$= \frac{2(2x+10)^2-4(2x^2+20x+2)}{(2x+10)^3}$$

$$= \frac{8x^2+80x+200-8x^2-80x-8}{(2x+10)^3}$$

$$= \frac{192}{8(x+5)^3}$$

$$\mathbf{f''(x) = \frac{24}{(x+5)^3}}$$

c First derivative

$$y' = 3$$

Second derivative

$$\mathbf{y'' = 0}$$

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d First derivative

$$f'(x)=\frac{9}{4}x^{\frac{1}{2}}$$

Second derivative

$$f''(x)=\frac{9}{8}x^{-\frac{1}{2}}$$

$$\mathbf{=\frac{9}{8x^{\frac{1}{2}}}}$$

e First derivative using product rule and chain rule

$$y=(2x+1)\left(x^2+3\right)^3$$

$$y'=(2x+1)\left(6x\left(x^2+3\right)^2\right)+2\left(x^2+3\right)^3$$

$$=\left(x^2+3\right)^2\left(6x(2x+1)+2\left(x^2+3\right)\right)$$

$$=\left(x^2+3\right)^2\left(14x^2+6x+6\right)$$

Second derivative using product rule and chain rule

$$y''=\left(x^2+3\right)^2(28x+6)+\left(14x^2+6x+6\right)\left(4x\left(x^2+3\right)\right)$$

$$=\left(x^2+3\right)\left(\left(x^2+3\right)(28x+6)+\left(14x^2+6x+3\right)(4x)\right)$$

$$=\left(x^2+3\right)\left(28x^3+6x^2+84x+18+56x^3+24x^2+12x\right)$$

$$\mathbf{=\left(x^2+3\right)\left(84x^3+30x^2+96x+18\right)}$$

f First derivative

$$y'=0$$

Second derivative

$$\mathbf{y''=0}$$

Question 4

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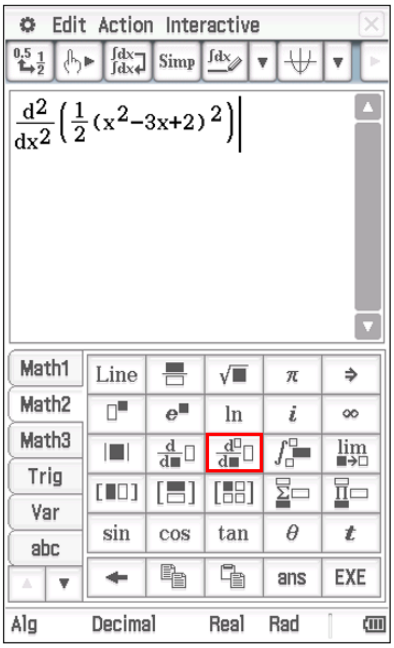

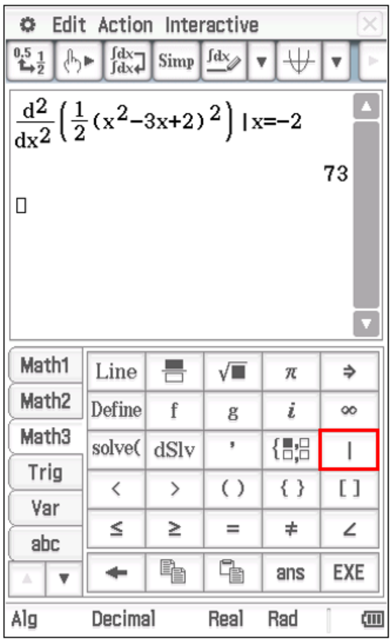

1 Open the Keyboard and tap Math2.

2 Enter the function.

3 Tap on the nth derivative template.

4 Enter x and 2 into the template.

5 Move the cursor to the end of the expression.

6 Tap Math3.

7 Tap the | symbol.

8 Enter x = -2 and press ENTER.

Alternatively, tap Interactive > Calculation diff > Derivative at value. Complete the fields in the dialogue box and change the Order field to 2.

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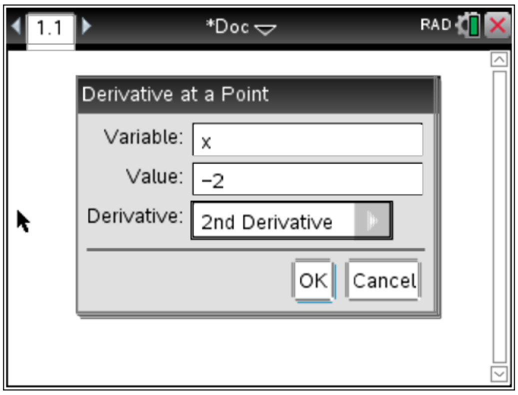
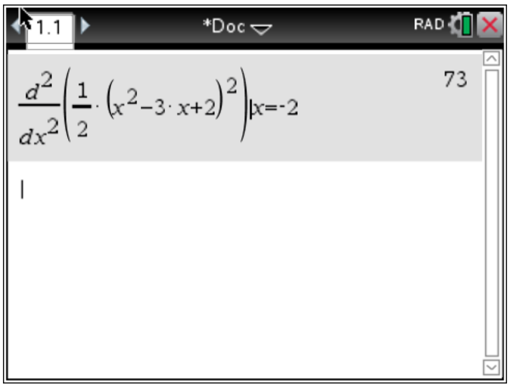


1 Press menu > Calculus > Derivative at a Point to open the dialogue box.

2 For the Value: field, enter -2.

3 Change the Derivative: field to 2nd Derivative.

4 Press OK.

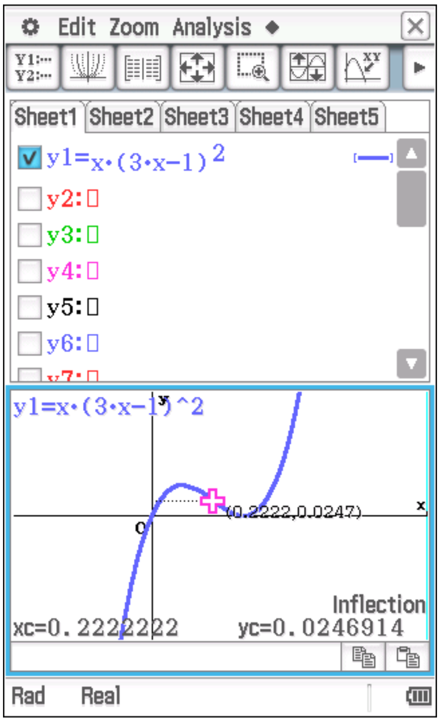


5 In the derivative template, enter the function.

6 Press enter.

Question 5

ClassPad

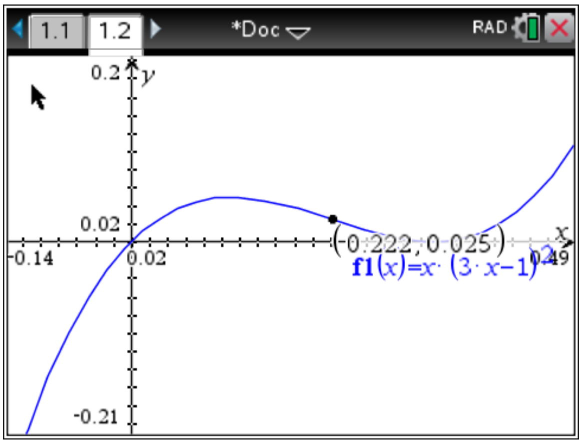


1 Graph the function and adjust the window settings if needed.

2 Tap Analysis>G-Solve>Inflection.

3 The graph is concave down when x is less than 0.22 and concave up when x is greater than 0.22.

TI-Nspire



1 Graph the equation and adjust the window settings if necessary.

2 Press menu > Analyze Graph > Inflection.

3 When prompted for the lower bound?, move the cursor to the concave down section and press enter.

4 When prompted for the upper bound?, move the cursor to the concave up section and press enter.

5 The coordinates of inflection will appear on the screen.

The function is concave down for $x < 0.22$, concave down for $x > 0.22$.

The point of inflection is (0.22, 0.02).

Question 6 (2 marks)

(✓ = 1 mark)

Some examples.

$f(x) = k, f'(x) = 0, f''(x) = 0$, where k is a constant.

$f(x) = e^x, f'(x) = e^x, f''(x) = e^x$

Appropriate function✓

Show that $f'(x) = f''(x)$✓

Question 7 (1 mark)

(✓ = 1 mark)

The function is a cubic, so there is only **one point of inflection**.✓

Question 8 (4 marks)

(✓ = 1 mark)

$f'(x) = 6x^2 + 8ax - 2bx + 2a^2 - 2ab$

$\boldsymbol{f'(2) = 24 + 16a - 4b + 2a^2 - 2ab = 24}$ ✓

$\boldsymbol{f''(x) = 12x + 8a - 2b}$ ✓

$\boldsymbol{f''(2) = 24 + 8a - 2b = 26}$ ✓

Solve the simultaneous equations for *a* and *b*.

$\boldsymbol{a = 1, b = 3}$ ✓

Question 9 (3 marks)

(✓ = 1 mark)

$$\boldsymbol{\frac{dy}{dx} = \frac{2}{(x^2+2)^{\frac{3}{2}}}, \quad \frac{d^2y}{dx^2} = \frac{-6x}{(x^2+2)^{\frac{5}{2}}}}$$ ✓

$$\frac{dy}{dx} + \frac{d^2y}{dx^2} = 0$$

$$\frac{2}{(x^2+2)^{\frac{3}{2}}} + \frac{-6x}{(x^2+2)^{\frac{5}{2}}} = 0$$

$$2 + \frac{-6x}{x^2+2} = 0$$

$\boldsymbol{x^2 - 3x + 2 = 0}$

$\boldsymbol{(x-1)(x-2) = 0}$ ✓

$\boldsymbol{x = 1, x = 2}$ ✓

EXERCISE 2.3 Stationary points

Question 1

$$f(x)=\frac{x^2}{x+1}$$

$$f'(x)=\frac{(x+1)\times(2x)-x^2\times 1}{(x+1)^2}=\frac{x^2+2x}{(x+1)^2}$$

$$f''(x)=\frac{(x+1)^2(2x+2)-2(x+1)(x^2+2x)}{(x+1)^4}$$

$$f''(2)=\frac{(3)^2(6)-2(3)(8)}{81}=\frac{6}{81}$$

$$\boldsymbol{f''(2)=\frac{2}{27}}$$

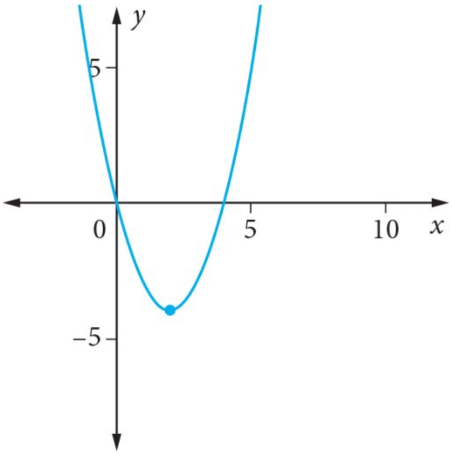
Question 2

The function is a cubic, so there is one point of inflection.

Look at where the concavity changes from concave down to concave up. This is approximately at **(−0.6, 10.6).**

Question 3

Sketch the graph of $f(x) = x^2 - 4x$.



Find $f'(x)$ and solve for zero.

$f'(x) = 2x - 4 = 0$

$2(x - 2) = 0$

$x = 2$

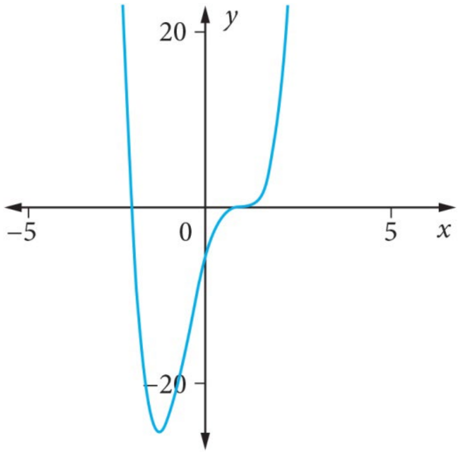
Around $x = 2$, gradient changes from negative to positive, which is a turning point.

$f(2) = (2)^2 - 4(2) = -4$

The coordinates are **(2, –4)**, which is a **local minimum**.

Question 4

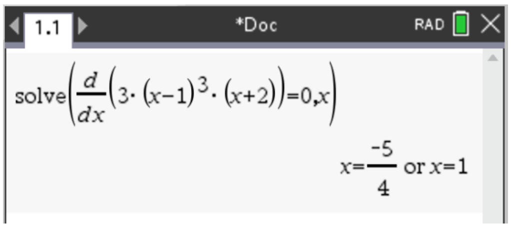
Sketch the graph of $f(x) = 3(x-1)^3(x+2)$.

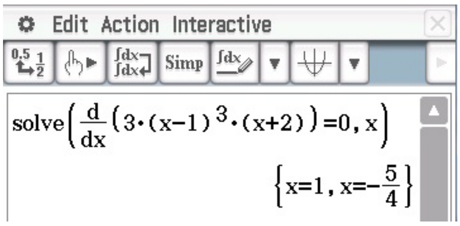


Find $f'(x)$ and solve for zero.

$$f'(x) = 3\left[(x-1)^3 \times 1 + (x+2) \times 3(x-1)^2\right] = 0$$

$$3(x-1)^2\left[(x-1) + 3x + 6\right] = 3(x-1)^2(4x+5) = 0$$





$f'(x) = 0$ for $x = 1$ or $x = -\dfrac{5}{4}$

Check to see if the sign of the gradient changes or stays the same on either side to identify the nature of the stationary points.

Around $x = -\dfrac{5}{4}$, gradient changes from negative to positive.

Around $x = 1$, gradient remains positive.

Find the coordinates of the stationary point of inflection.

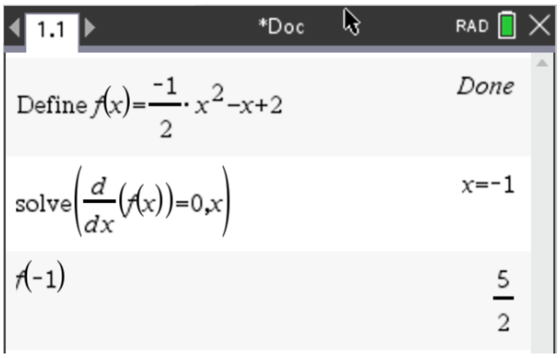
The stationary point of inflection is at $x = 1$.

$f(1) = 0$

The stationary point of inflection is at **(1, 0)**.

Question 5

ClassPad

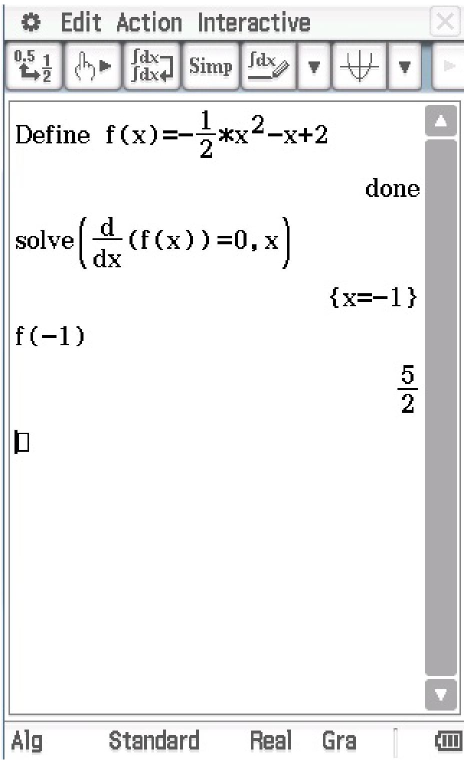


1 Define and highlight ***f*(*x*)** as shown above.

2 Derive using **Interactive > Diff**, and equate to zero, then solve using **Equation/Inequality** to solve for *x*.

3 Substitute the *x*-coordinates into the defined function to determine the corresponding *y*-coordinates of the turning points.

TI-Nspire



1 Define ***f*(*x*)** as shown above.

2 Use the **derivative** template to set the derivative of ***f*(*x*) = 0** and solve for ***x***.

3 Substitute the *x*-coordinates into the function to determine the corresponding *y*-coordinates of the turning points.

4 Graph **f(x)**, which shows the local maximum and minimum.

5 Adjust the window settings to suit.

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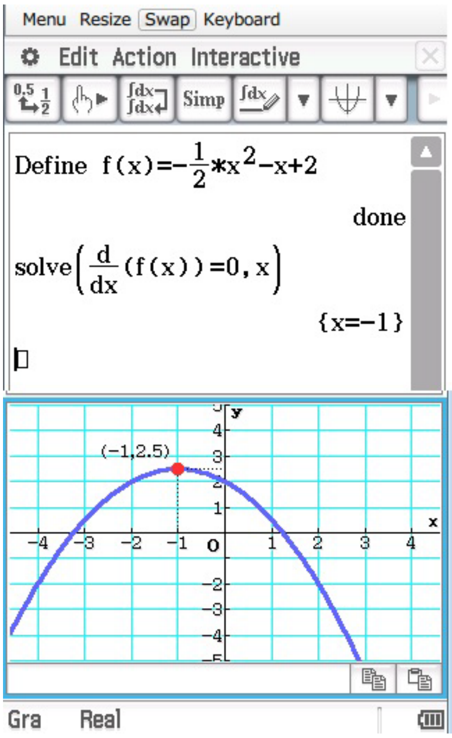
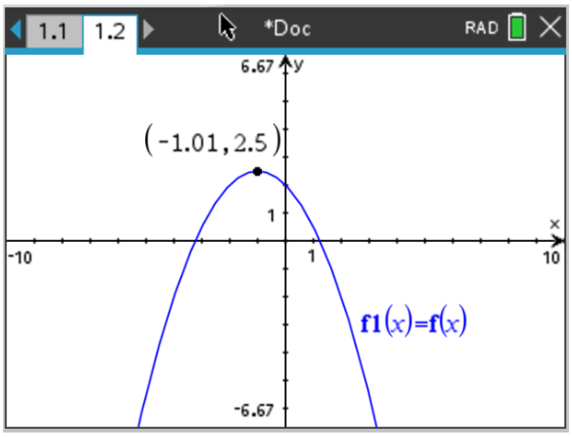


4 Graph ***f*(*x*)** by dragging into the graph screen.

5 Adjust the window settings to suit.

6 Tap **Analysis** > **G-Solve** > **Max** to display the approximate coordinates of the local maximum.



6 To confirm the turning points, press **menu** > **Geometry** > **Points & Lines** > **Point On.**

7 Press **esc** to remove the point tool.

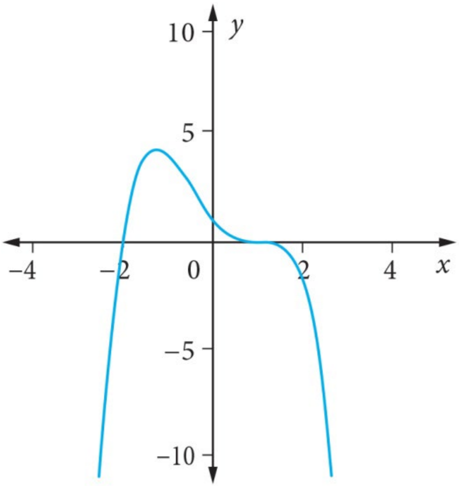
8 Click twice on the *x*-coordinate of the point and enter **–1**.

9 The exact value of the corresponding *y*-coordinate will be displayed.

The turning point is at $\left(-1, \frac{5}{2}\right)$ and is a local maximum.

Question 6

Sketch the graph of $y=-\frac{1}{2}x^4+\frac{1}{2}x^3+\frac{3}{2}x^2-\frac{5}{2}x+1$.



$y'=0$ for $x = 1$ and $x=-\frac{5}{4}$

Check to see if the sign of the gradient changes or stays the same on either side to identify the nature of the stationary points.

Around $x=-\frac{5}{4}$, gradient changes from positive to negative. **The coordinates of the local maximum are** $\left(-\frac{5}{4},\frac{2187}{512}\right)$.

The stationary point of inflection is at $x = 1$.

$f(1) = 0$

The stationary point of inflection is at (1, 0).

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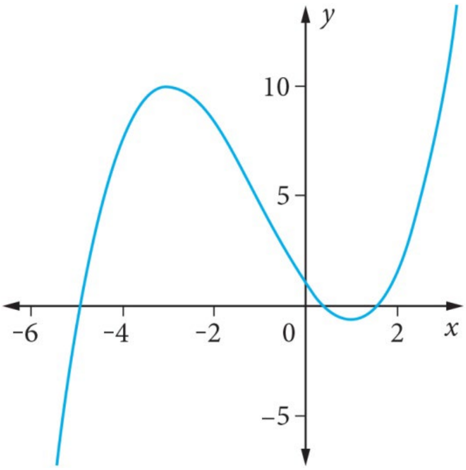
Question 7 (4 marks)

(✓ = 1 mark)

Sketch the graph of $y=\frac{1}{3}x^3+x^2-3x+1$.



Find y' and solve for zero.

$y=\frac{1}{3}x^3+x^2-3x+1$

$\boldsymbol{y'=x^2+2x-3=0}$ ✓

$\boldsymbol{(x+3)(x-1)=0}$

$\boldsymbol{x=-3,\ x=1}$ ✓

Around $x=-3$, gradient changes from positive to negative.

Around $x = 1$, gradient changes from negative to positive.

Both are turning points.

Find the coordinates and nature of the turning points.

$\boldsymbol{f(-3)=10}$ **and** $\boldsymbol{f(1)=-\frac{2}{3}}$ ✓

Therefore, the coordinates are $\boldsymbol{(-3, 10)}$ **and** $\boldsymbol{\left(1,-\frac{2}{3}\right)}$**, and are both local minimum.**✓

Question 8 (2 marks)

(✓ = 1 mark)

The gradient has the rule $\boldsymbol{y' = 3(x - 1)^2}$.✓

When $x = 1$, $y' = 0$, indicating that the function has a stationary point at $x = 1$.

y' is positive for all $x \in R\backslash\{1\}$, indicating that the gradient of y is positive to both the left and right of $x = 1$. **Hence (1, 1) is a stationary point of inflection.**✓

Note: The same result could be achieved by noting that the given cubic is $y = x^3$ translated 1 unit right and 1 unit up.

Question 9 (1 mark)

(✓ = 1 mark)

f has a stationary point at $x = 3$.

As the gradient of f has the same sign on both sides of this stationary point, it cannot be either a local maximum or minimum; $\boldsymbol{x = 3}$ **is a stationary point of inflection.**✓

Question 10 (2 marks)

(✓ = 1 mark)

As $f(x)$ is a cubic function, f' is a quadratic function.

As f' has a maximum value of 10, the graph of f' is a downwards opening parabola.

Given $f'(-3) = f'(2) = 0$, $f(x)$ has stationary points at $x = -3$ and $x = 2$.

For $x < -3$, $f'(x) < 0$. For $-3 < x < 2$, $f'(x) > 0$. For $x > 2$, $f'(x) < 0$.

This shows that $\boldsymbol{f}$ **has a local minimum at** $\boldsymbol{x = -3}$✓ and **a local maximum at** $\boldsymbol{x = 2}$.✓

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Question 11 (6 marks)

(✓ = 1 mark)

a $f'(x) = 6x - 3x^2 = 3x(2 - x)$.✓

For $f'(x) = 0$, $x = 0$ and $x = 2$.

$f(0) = 0$ and $f(2) = 4$

Stationary points at (0, 0) and (2, 4).

$f''(x) = 6 - 6x$ ✓

$f''(0) > 0$ and $f''(2) < 0$.

Hence the stationary points are not stationary points of inflection.

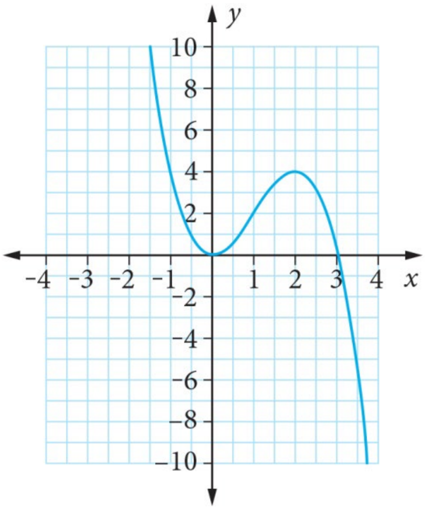
For (0, 0), $x < 0, f'(x) < 0$ and for $x > 0, f'(x) > 0$.

Hence **(0, 0) is a local minimum**.✓

For (2, 4), $x < 2, f'(x) > 0$ and for $x > 2, f'(x) < 0$.

Hence **(2, 4) is a local maximum**. ✓

b



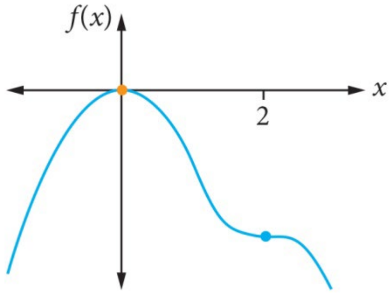
show correct general shape✓

accurately show intercepts and turning points✓

Question 12 (5 marks)

(✓ = 1 mark)

The graph shown below displays the required properties.



a **False.**✓

$f'(0) = 0$ and $f''(0) = 0$, but the gradients are not the same on either side of $x = 0$.

b **False.**✓

$f'(2) = 0$ and $f''(2) = 0$, but the gradients are not the same on either side of $x = 2$.

c **True.**✓

$f'(2) = 0$ and $f''(2) = 0$, and the gradients on either side of $x = 2$ are negative.

d **True.**✓

$f'(0) = 0$, $f'(x) > 0$ for $x < 0$ $f'(x) < 0$ for $x > 0$.

e **False.**✓

There is a stationary point of inflection at x =2 so the function is not a turning point. Hence it cannot be a local minimum or a local maximum.

Question 13 (3 marks)

(✓ = 1 mark)

$\boldsymbol{f'(x) = 3x^2 + 2ax + b}$ ✓

$\boldsymbol{f'(1) = 0 \Rightarrow 2a + b = -3}$ **and** $\boldsymbol{f'(-3) = 0 \Rightarrow -6a + b = -27}$ ✓

Solve the simultaneous equations to find a and b.

$\boldsymbol{a = 3, b = -9}$ ✓

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Question 14 (6 marks)

(✓ = 1 mark)

a

$$y = 2x^{\frac{1}{2}} - x^2 + x \checkmark$$

$$\frac{dy}{dx} = \frac{1}{2} \times 2x^{-\frac{1}{2}} - 2x + 1 = \frac{1}{\sqrt{x}} - 2x + 1 \checkmark$$

b

At $x = 1$, $\frac{dy}{dx} = \frac{1}{1} - 2 + 1 = 0$ ✓

The gradient is zero, so there is a stationary point at $x = 1$.✓

c

$$\frac{d^2y}{dx^2} = -\frac{1}{2}x^{-\frac{3}{2}} - 2 = -\frac{1}{2x^{\frac{3}{2}}} - 2 \checkmark$$

$x = 1$, $\frac{d^2y}{dx^2} < 0$, so there is no point of inflection.

For $x < 1$, $\frac{dy}{dx} > 0$ and for $x > 1$, $\frac{dy}{dx} < 0$.

Hence at $x = 1$ there is a local maximum. ✓

Question 15 (2 marks)

(✓ = 1 mark)

$$f'(x) = \frac{1}{2\sqrt{x}} + 2x$$

For stationary points to exist, $f'(x) = 0$ for some x.

This is not possible, because $x > 0$✓, so **we want the sum of two positive quantities $\frac{1}{2\sqrt{x}}$ and $2x$ to equal zero, which is not possible.**✓

Question 16 (2 marks)

(✓ = 1 mark)

As the question asks about stationary points of the function $f(x) = ax^3 - bx^2 + cx$, its derivative is required. **The function has derivative $f'(x) = 3ax^2 - 2bx + c$ and stationary points when $f'(x) = 0$.**✓

The function f has no stationary points when $f'(x) = 0$ has no real solutions, that is, when the quadratic equation $3ax^2 - 2bx + c = 0$ has no real solutions. This occurs when the discriminant of the quadratic is negative, which is when $(-2b)^2 - 4(3a)(c) < 0$. This inequality is simplified to give the condition $4b^2 - 12ac < 0$, or equivalently, $\boldsymbol{c > \frac{b^2}{3a}}$.✓

Question 17 (2 marks)

(✓ = 1 mark)

a For $f(x) = 4x^3 + 5x - 9$, $\boldsymbol{f'(x) = 12x^2 + 5}$.✓

b **As $x^2 \geq 0$ for all x, $f'(x) \geq 5$ for all x.**✓

Question 18 (2 marks)

(✓ = 1 mark)

As $p(x)$ is a cubic, its derivative is a quadratic polynomial.

This has at most two real roots, and **hence $p(x)$ has at most two stationary points.**✓

Thus, possible values for m are **0, 1 and 2**.✓

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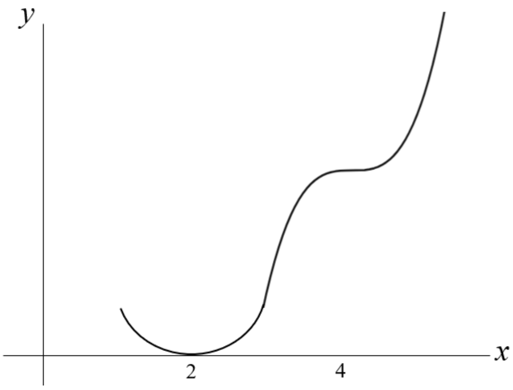
EXERCISE 2.4 Curve sketching

Question 1

The graph below satisfies the given conditions.



There is no local maximum at and $x = 4$ and no local maximum at $x = 2$, so options A, C and D are False.

The sign of the gradients on either side of $x = 2$ are different, so there is no point of inflection at $x = 2$. Hence E is False.

$x = 4$ is a stationary point and the gradient is positive on either side of $x = 4$. Hence this is a point of inflection.

The answer is B.

Question 2

$$f'(x) = 5x^4$$
$$f'(x) = 0 \Rightarrow x = 0$$
$$f(0) = -1$$

The gradient is positive on either side of $x = 0$, so **(0, –1) is a stationary point of inflection.**

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Question 3

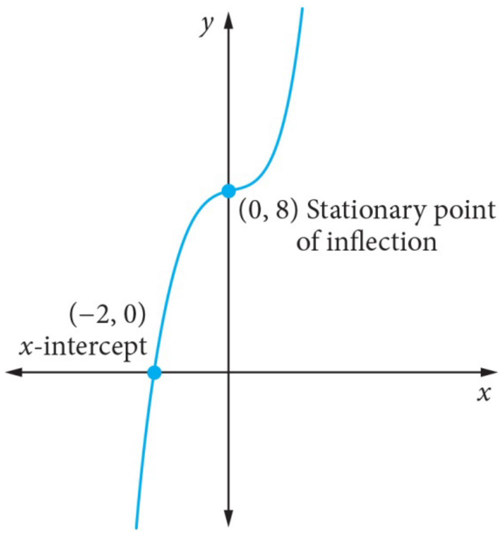
The graph of $f(x) = x^3 + 8$ is a vertical translation of the graph of the cubic $f(x) = x^3$.

The y-intercept is $f(0) = 8$ with co-ordinates (0, 8).

The x-intercept is the solution to $f(x) = 0$. $x^3 + 8 = 0 \Rightarrow x = \sqrt[3]{-8} = -2$, with co-ordinates (–2, 0).

$f'(x) = 3x^2 = 0 \Rightarrow x = 0$, $f(0) = 8$

$f'(x) > 0$ on either side of $x = 0$, so (0, 8) is also a stationary point of inflection.



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Question 4

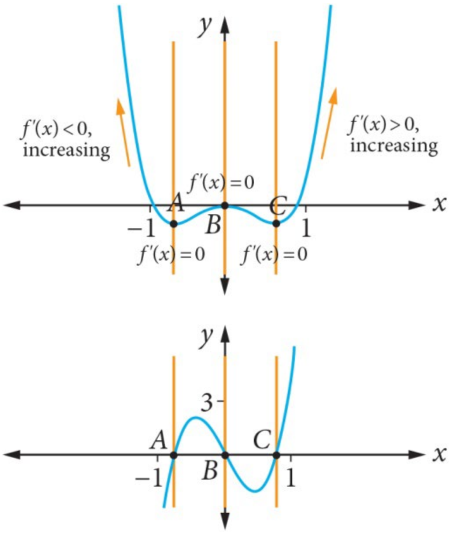
There are three stationary points, A (–0.6, –2), B (0, 0), C (0.6, –2). (co-ordinates for A and C are estimates).

For $x < -0.6$ the gradient is negative and increasing in magnitude.

For $x > 0.6$ the gradient is positive and increasing in magnitude.

From A to B, the gradient is positive, increasing to a maximum and then decreasing to zero.

From B to C, the gradient is negative, decreasing to a minimum and then increasing to zero.



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Question 5

a The x-intercepts occur when $-x(x+2)(x-3)=0$, so $x = 0$, $x = -2$, $x = 3$.

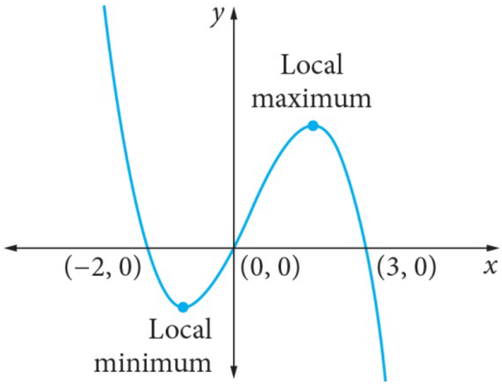
$f(0) = 0$. The y-intercept is 0.

The shape of the graph is cubic with three x-intercepts, so there are two turning points (stationary points).

For $x < -2$, $f(x) > 0$ and for $x > 3$, $f(x) < 0$.

For $-2 < x < 0$, $f(x) < 0$, so there is a local minimum between $x = -2$ and $x = 0$.

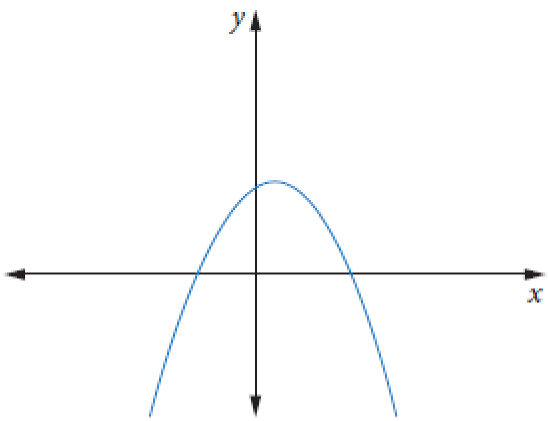
For $0 < x < 3$, $f(x) > 0$, so there is a local maximum between $x = 0$ and $x = 3$.



b The local minimum and the local maximum are the x-intercepts of the derivative graph.

The gradient is negative to the left of the local minimum and to the right of the local maximum.

The gradient is positive between the local minimum and the local maximum, reaching a maximum value between $x = 0$ and the local maximum.



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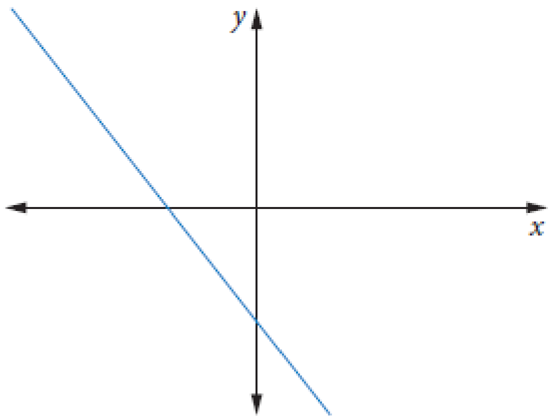
Question 6 (2 marks)

(✓ = 1 mark)

The function given is a quadratic, so the derivative function will be linear, hence the graph will be a straight line.

There is one stationary point where the *x*-value is negative, which is the turning point of the parabola. To the left of the stationary point the gradient is positive, and to the right of the stationary point the gradient is negative.✓



✓

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Question 7 (5 marks)

(✓ = 1 mark)

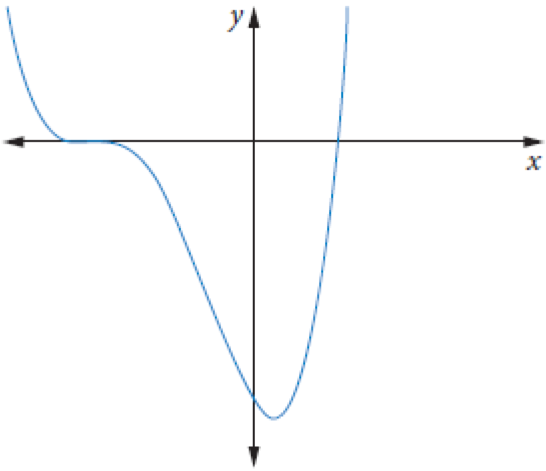
$f'(-2) = f''(-2) = 0$ means there is a **stationary point of inflection at $x = -2$.**✓

Since $f(-2) = 0$, the co-ordinates of the **stationary point of inflection is at (–2, 0)**.✓

Also, to the left of the stationary point of inflection the graph has a negative slope.

The function has exactly two stationary points. One of these is the stationary point of inflection, so there is another **stationary point at $x = \frac{1}{4}$.**✓

For $x > \frac{1}{4}$ the gradient is positive.



correctly show the stationary point of inflection✓

correctly show the turning point✓

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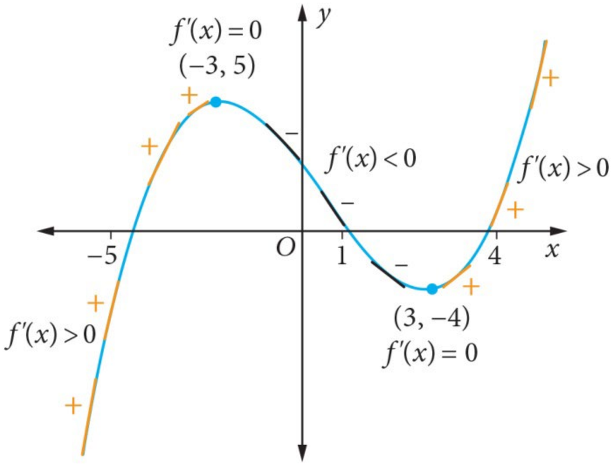
Question 8 (2 marks)

(✓ = 1 mark)

a



$f'(x)$ is negative in the section of the curve between the local maximum (–3, 5) and the local minimum (3, –4). **This is in the interval –3 < *x* < 3.**✓

b **(3, –4)**✓

$f''(x) = 0$ is the positive *x*-coordinate, *X*, of the turning point of the graph of the parabola given by $f'(x)$, which is concave up. The function *f*(*x*) is a cubic, so $f''(x)$ is linear. Hence for all points greater than *X*, $f''(x) > 0$.

We can also show this algebraically.

$f(x) = ax^3 + bx^2 + cx + d$, where $a > 0$.

$f'(x) = 3ax^2 + 2bx + c$

$f'(x) = 0 \Rightarrow 3ax^2 + 2bx + c = 0$

Solving for *x* gives $6ax + 2b = \sqrt{4b^2 - 4ac}$ for the turning point with positive *x*.

$f''(x) = 6ax + 2b$

So $f''(x) = \sqrt{4b^2 - 4ac}$ which is positive.

A reasonable attempt using either one of the above two explanations. ✓

EXERCISE 2.5 Straight line motion

Question 1

It is given that $f(x) = x^2 + \frac{p}{x}, x \neq 0$.

The stationary points of *f* occur for $f'(x) = 0$.

$$f'(x) = 2x - \frac{p}{x^2}$$

$$\frac{2x^3 - p}{x^2} = 0$$

$$2x^3 - p = 0$$

There is a stationary point on the graph of *f* when $x = 2$.

$$\therefore 2(2)^3 - p = 0$$

$$p = 16$$

The correct response is **E**.

Question 2

For $x < -3$ the gradient is positive, so all of the corresponding *y* values in the graph of the derivative function will be positive.

This is not the case for options A, C, D and E.

The answer is B.

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Question 3

a Differentiate to find the velocity.

$v = x'(t)$

$\boldsymbol{v = 4t}$

b Differentiate the velocity equation.

$a = v'(t)$

$a = 4$

$\boldsymbol{a(2) = 4 \text{ m/s}^2}$

Question 4

a $h(2) = 1.5(2) + (2)^2 - 0.5(2)^3 = 3$

The height is 3 m

$v(t) = h'(t) = 1.5 + 2t - 1.5t^2$

$v(2) = 1.5 + 2(2) - 1.5(2)^2 = -0.5$

The velocity is –0.5 m/s

$a(t) = v'(t) = 2 - 3t$

$a(2) = 2 - 3(2) = -4$

The acceleration is –4 m/s²

b The initial velocity is $v(0) = 1.5$ m/s[1]. (a positive value)

After 2 seconds the velocity is negative. **This means the object has reached its maximum height and is now falling with a speed of 0.5 m/s**.

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Question 5

a $v(t) = x'(t) = 6t + 2$

$v(t) = 6t + 2$

b $a(t) = v'(t) = 6$

$a(6) = 6$

The acceleration is 6 m/s^2

Question 6 (6 marks)

(✓ = 1 mark)

a $v(t) = x'(t) = 3t^2 + 12t - 2$

$v(t) = 3t^2 + 12t - 2$ ✓

$a(t) = v'(t) = 6t + 12$ ✓

b $x(5) = (5)^3 + 6(5)^2 - 2(5) + 1 = 266$

The displacement after 5 seconds is 266 m✓

c $v(5) = 3(5)^2 + 12(5) - 2 = 133$

The velocity after 5 seconds is 133 m/s✓

d $a(t) = 6t + 12$

$a(0) = 12$

The initial acceleration is 12 m/s^2✓

e $a(5) = 6(5) + 12 = 42$

The acceleration after 5 seconds is 42 m/s^2✓

Question 7 (7 marks)

(✓ = 1 mark)

a $x(4) = (4)^3 - 5(4)^2 + 6(4) + 10 = 18$

$x(2) = (2)^3 - 5(2)^2 + 6(2) + 10 = 10$

Average velocity is average rate of displacement.

$\frac{18-10}{4-2} = 4$

The average rate of change is 4 m/s✓

b $v(t) = x'(t) = 3t^2 - 10t + 6$

$v(2) = 3(2)^2 - 10(2) + 6 = -2$

The velocity at 2 seconds is –2 m/s✓

c $v(4) = 3(4)^2 - 10(4) + 6 = 14$

The velocity at 4 seconds is 14 m/s✓

d Average of the velocities = $\frac{14+(-2)}{2} = 6$

The average of the two velocities is **6 m/s.**✓

This is not the same value as using $\frac{x(4)-x(2)}{4-2}$ **from part a.**✓

e $a(t) = v'(t) = 6t - 10$

$a(2) = 6(2) - 10 = 2$

The acceleration at 2 seconds is 2 m/s^2✓

f $a(5) = 6(5) - 10 = 20$

The acceleration at 2 seconds is 20 m/s^2✓

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Question 8 (9 marks)

(✓ = 1 mark)

a $v(t) = s'(t) = 4t - 8$

$v(0) = 4(0) - 8 = -8$

The initial velocity is –8 m/s✓

b $a(t) = v'(t) = 4$

The acceleration is constant because its value is 4 m/s^2 for all values of *t*.✓

c $s(5) = 2(5)^2 - 8(5) + 3 = 13$

The displacement after 5 seconds is 13 m✓

d The particle will be at rest when its velocity is zero.

$v(t) = 4t - 8 = 0$

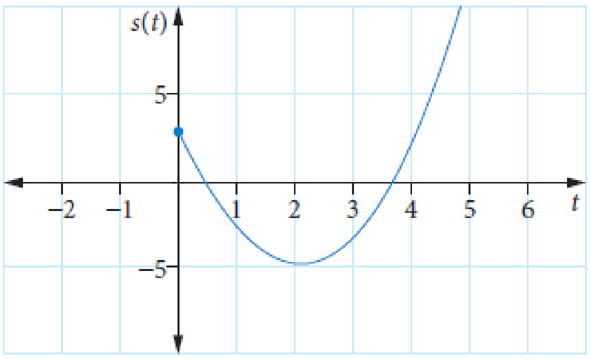
$t = 2$

The particle will be at rest after 2 seconds.✓

e $s(2) = 2(2)^2 - 8(2) + 3 = -5$

The displacement after 2 seconds will be –5 m✓

f



✓

$s(t) = 2t^2 - 8t + 3$**. The equation for displacement is a quadratic, so its graph is a parabola. The *s* intercept is *s*(0) = 3. The turning point is when** $s'(t) = 0$**. This occurs at *t* = 2, and *s*(2) = –5. The turning point is at (2, –5) and *s*(5) =13.✓**

or

Start at 3 on the vertical axis. This is the initial position of the particle from a fixed point. The particle moves towards the fixed point and passes it, and at *t* = 2 it is 5 metres behind the fixed point. (This is the turning point of the parabola).

The particle changes direction and moves towards the fixed point. After 5 seconds it 13 m in front of the fixed point.✓

Question 9 (4 marks)

(✓ = 1 mark)

a $\boldsymbol{v(t) = x'(t) = 20(t+1)}$ ✓

$v(5) = 20(5+1) = 120$

The velocity after 5 hours is 120 km/h✓

b $\boldsymbol{a(t) = v'(t) = 20}$ ✓

$a(5) = 20$

The acceleration after 5 hours is 20 km/h^2✓

Question 10 (2 marks)

(✓ = 1 mark)

$\boldsymbol{a(t) = v'(t) = 4t}$ ✓

$a(5) = 4(5) = 20$

The acceleration after 5 seconds is 20 m/s^2✓

Question 11 (2 marks)

(✓ = 1 mark)

$\boldsymbol{a(t) = v'(t) = 8t - 4}$ ✓

$a(1) = 8(1) - 4 = 4$

The acceleration after 1 second is 4 m/s^2✓

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Question 12 (4 marks)

(✓ = 1 mark)

a $v(t) = x'(t) = \frac{1}{(2t+5)^2}$ ✓ (Can use quotient rule)

b $a(t) = v'(t) = -\frac{4}{(2t+5)^3}$ ✓ (Can use quotient rule)

c $a(t) = -4 \times \frac{1}{(2t+5)^3} = -4v(t)$ ✓

The magnitude of $-4v(t)$ is $4v(t)$✓, so the magnitude of the acceleration is four times the magnitude of the velocity.

Question 13 (5 marks)

(✓ = 1 mark)

$v(t) = 2pt + q, \quad a(t) = 2p$ ✓

$a(3) = -4$, $2p = -4$✓, so $p = -2$

$v(3) = -24$,

$2p(3) + q = -24$✓

$-12 + q = -24$, so $q = -12$

$x(3) = -34$,

$p(3)^2 + q(3) + r = -34$✓

$-18 - 36 + r = -34$

$r = 20$

Thus **$p = -2$, $q = -12$, $r = 20$**✓

EXERCISE 2.6 Optimisation problems

Question 1

Differentiate twice.

velocity $v(t) = x'(t) = \dfrac{2t^2 + 4t + 1}{\left(2t^2 - 1\right)^2}$.

acceleration $a(t) = v'(t) = \dfrac{-\left(8t^3 + 24t^2 + 12t + 4\right)}{\left(2t^2 - 1\right)^3}$

$$a(2) = \frac{-\left(8(2)^3 + 24(2)^2 + 12(2) + 4\right)}{\left(2(2)^2 - 1\right)^3}$$

$$= -\frac{188}{343}$$

$$\mathbf{\approx -0.55}$$

Question 2

Find the t value when the velocity is zero.

Differentiate to find velocity.

velocity $v(t) = x'(t) = 6\left(t^2 + 2t - 3\right)$. (Expand and differentiate, or use the calculator)

$6\left(t^2 + 2t - 3\right) = 0$

$(t-1)(t+3) = 0$ (or use the SOLVE function with a calculator)

$t = 1, t = -3$

Since time is positive, $t = 1$

The object is at rest after 1 second.

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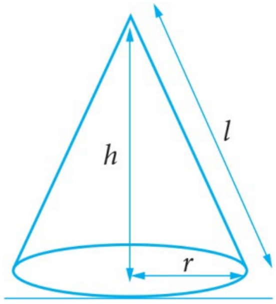
Question 3

Sketch a diagram.



Height = h cm and radius = r cm; $h = 2 - r$ cm

The volume V of a right circular code with height h and radius r is given by

$$V = \frac{1}{3}\pi r^2 h$$

$$V(r) = \frac{1}{3}\pi r^2(2 - r) = \frac{1}{3}\pi(2r^2 - r^3)$$

Solve $V'(r) = 0$.

$$\frac{d}{dr}V(r) = \frac{1}{3}\pi(4r - 3r^2) = 0$$

$$r = 0 \text{ or } r = \frac{4}{3}$$

The solution $r = 0$ is not physically realistic. So, $r = \frac{4}{3}$.

$$V\left(\frac{4}{3}\right) = \frac{1}{3}\pi\left(2\left(\frac{4}{3}\right)^2 - \left(\frac{4}{3}\right)^3\right) \approx 1.24$$

The maximum volume of the cone is **1.24 cm^3**.

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Question 4

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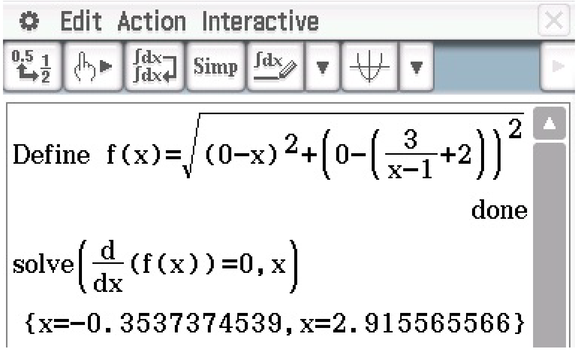
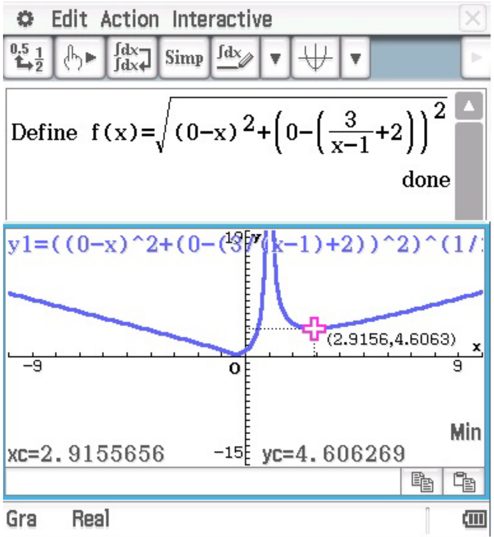


1 Define **f(x)** as shown above.

2 Derive using **Interactive > Calculation > Diff,** then using **Equation/Inequality,** solve for x.



3 Graph **f(x)**.

4 Adjust the window settings so that **ymax = 15** to view the local minimums and maximum.

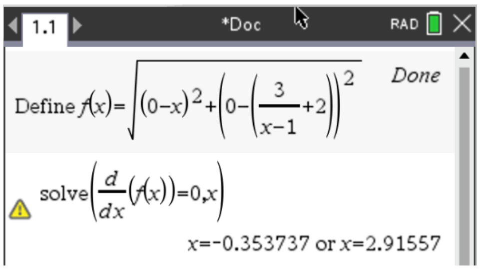
5 Tap **Analysis > G-Solve > Min**.

6 Press the **right arrow** key to move the cursor to the first local minimum.

7 Copy -0.3537374539 and calculate **f(-0.3537374539).**

8 The minimum distance **0.4145163 …** will be displayed.

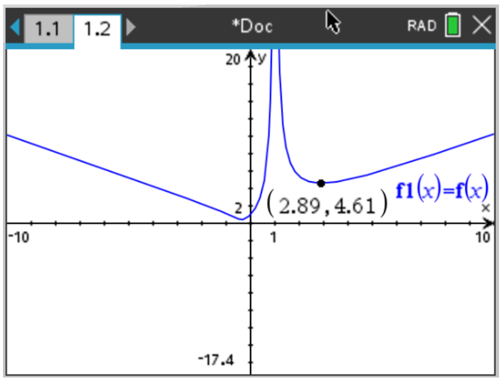
TI-Nspire



1 Define **f(x)** as shown above.

2 Use the **derivative** template to set the derivative of **f(x) = 0** and solve for **x**.

3 Press **ctrl + enter** for the approximate solutions.



4 Graph **f(x)**.

5 Press **menu > Window / Zoom > Zoom – Fit** to view the local minimums and maximum.

6 Press **menu > Analyze Graph > Minimum**.

7 When prompted for the **lower bound**, click to the left of the first minimum at $x = -0.35$.

8 When prompted for the **upper bound**, click to the right of the minimum point.

9 The coordinates of the local minimum will be displayed.

10 Copy 0.353737 and calculate **f(-0.353737).**

11 The minimum distance **0.4145163 …** will be displayed.

The minimum distance from the function $y=\frac{3}{x-1}+2$ to the point (0, 0) is **0.41**.

Question 5 (4 marks)

(✓ = 1 mark)

Volume is $\boldsymbol{V = 4x(4-x) = 16x - 4x^2}$ ✓

For maximum volume, $\boldsymbol{\frac{dV}{dx} = 0}$ ✓

$\frac{dV}{dx} = 16 - 8x$

$\frac{dV}{dx} = 0 \Rightarrow 16 - 8x = 0$ so $\boldsymbol{x = 2}$✓

$V = 4(2)(4-2) = 16$

The maximum volume is 16 cm^3✓

Question 6 (5 marks)

(✓ = 1 mark)

Let x be the length of the side of the base and y be the height of the prism.

$\boldsymbol{P = 8x + 4y}$✓

$y = \frac{P}{4} - 2x$

$V = x^2 y = \frac{Px^2}{4} - 2x^3$

For maximum volume, $\boldsymbol{\frac{dV}{dx} = 0}$ ✓

$\frac{dV}{dx} = \frac{Px}{2} - 6x^2$

$\boldsymbol{\frac{dV}{dx} = 0 \Rightarrow \frac{Px}{2} - 6x^2 = 0}$ ✓

Solve for x.

$x\left(\frac{P}{2} - 6x\right) = 0 \Rightarrow x = 0,\ x = \frac{P}{12}$

Since $x > 0$, $\boldsymbol{x = \frac{P}{12}}$ ✓

Hence $y = \frac{P}{4} - 2 \times \frac{P}{12} = \frac{P}{12}$

Since the base sides and the height are the same, the shape for maximum volume is a cube.✓

Question 7 (4 marks)

(✓ = 1 mark)

a $\boldsymbol{P = 4x + 4y + 4h}$ ✓

$P = 4x + 4y + 4h \Rightarrow h = \dfrac{P}{4} - x - y$

$V = xyh$

$= xy\left(\dfrac{P}{4} - x - y\right)$

$\boldsymbol{= \dfrac{1}{4}xy\left(P - 4x - 4y\right)}$ ✓

b $y = 2x$

$V = \dfrac{1}{4}x(2x)\left(P - 4x - 4(2x)\right)$

$\boldsymbol{= \dfrac{1}{2}x^2\left(P - 12x\right)}$ ✓

$\dfrac{dV}{dx} = Px - 18x^2$

$\dfrac{dV}{dx} = 0 \Rightarrow Px - 18x^2 = 0$

$x\left(P - 18x\right) = 0$

$x = 0,\ x = \dfrac{P}{18}$

Since $x > 0$, take $x = \dfrac{P}{18}$.

$V = \dfrac{1}{2}x^2\left(P - 12x\right)$

$\dfrac{1}{2}\left(\dfrac{P}{18}\right)^2\left(P - 12 \times \dfrac{P}{18}\right)$

$\boldsymbol{= \dfrac{1}{2}\left(\dfrac{P}{18}\right)^2\left(\dfrac{P}{3}\right)}$ ✓

$= \dfrac{1}{2} \times P^3 \times \dfrac{1}{18^2 \times 3}$

$= \dfrac{P^3}{6 \times 18^2}$

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Question 8 (4 marks)

(✓ = 1 mark)

a $S = x^2 + y^2 \Rightarrow \boldsymbol{y = \sqrt{S - x^2}}$ ✓

The product, P, of the two numbers is xy

Substitute for y in xy.

$\boldsymbol{P = xy = x\sqrt{S - x^2}}$ ✓

b Differentiate to obtain the maximum.

$P = x\sqrt{S - x^2}$

$$\frac{dP}{dx} = \sqrt{S - x^2} - \frac{x^2}{\sqrt{S - x^2}}$$ ✓

Solve for x.

$$\sqrt{S - x^2} - \frac{x^2}{\sqrt{S - x^2}} = 0$$ ✓

$$x = \sqrt{\frac{S}{2}} = \frac{\sqrt{S}}{\sqrt{2}}$$

Question 9 (7 marks)

(✓ = 1 mark)

a The volume of a prism is the product of the area of one end and its length.

The triangle has base x and height $\frac{\sqrt{3}}{2}x$, giving an area of $\frac{\sqrt{3}x^2}{4}$ cm^2.

$$\frac{\sqrt{3}}{4}x^2 \times y = 1000 \checkmark$$

$$y = \frac{4000}{\sqrt{3}x^2} = \frac{4000\sqrt{3}}{3x^2} \checkmark$$

b The surface area of the brick is the sum of the surface area of the triangular sides and the rectangular sides. The surface area of one triangular side was found to be $\frac{\sqrt{3}x^2}{4}$ cm^2 in part **a**, hence **the two triangular sides have area** $\frac{\sqrt{3}x^2}{2}$ **cm^2**.✓

The area of each rectangular side is xy.

Hence the total surface area is $A = 3xy + \frac{\sqrt{3}}{2}x^2$ cm^2

Using the expression for y in terms of x obtained in part **a**,

$$A(x) = 3x\left(\frac{4000\sqrt{3}}{3x^2}\right) + \frac{\sqrt{3}}{2}x^2 = \frac{4000\sqrt{3}}{x} + \frac{\sqrt{3}}{2}x^2 \text{ cm}^2. \checkmark$$

$$= \frac{4000\sqrt{3}}{x} + \frac{x^2\sqrt{3}}{2}$$

c $$\frac{dA}{dx} = -\frac{4000\sqrt{3}}{x^2} + \sqrt{3}x = \frac{\sqrt{3}}{x^2}\left(-4000 + x^3\right) = 0 \checkmark$$

$$x^3 = 4000 \checkmark$$

$$x = 10\sqrt[3]{4} \text{ or } 10 \times 4^{\frac{1}{3}} \text{ or } \sqrt[3]{4000} \checkmark$$

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Question 10 (3 marks)

(✓ = 1 mark)

The volume of the box is the product of its height, width and length, which is given by

$\boldsymbol{V(x) = x(6 - 2x)(8 - 2x)}$**, where** $\boldsymbol{0 < x < 3}$**.**✓

If this expression for volume has a local maximum, this will occur at some x value where

$\frac{dV}{dx} = 0$.

Differentiating the expression for $V(x)$ gives the condition $\boldsymbol{4(3x^2 - 14x + 12) = 0}$**.**✓

The quadratic formula gives these solutions as

$$x = \frac{-(-14) \pm \sqrt{(-14)^2 - 4(3)(12)}}{2(3)} = \frac{7 \pm \sqrt{13}}{3}.$$

These are approximately 1.13 and 3.54 to two decimal places, which shows that the larger solution for x can be disregarded as it is greater than 3. A sign diagram for $V'(x)$ shows that V has a local maximum at $x = \frac{7 - \sqrt{13}}{3}$.

Hence the x for which the volume of the box is maximized is the value closest to

$x = \frac{7 - \sqrt{13}}{3}$, which is **1.13 cm**✓

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Question 11 (4 marks)

(✓ = 1 mark)

Volume = length × width × height = $x(80-2x)(50-2x)$.✓

Set the derivative equal to 0 and solve.

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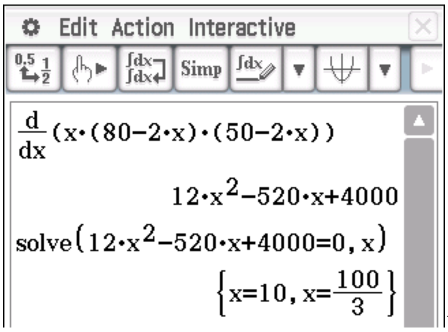

✓

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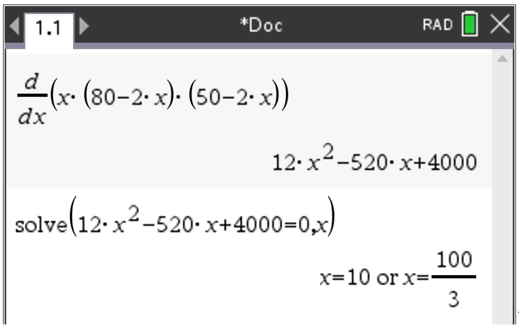

✓

Take $x = 10$ since $0 < x < 25$.✓

The volume will be maximum when length = 80 – 2x = 60 cm, width = 50 – 2x = 30 cm, height = 10 cm✓

Question 12 (5 marks)

(✓ = 1 mark)

a $h = \frac{10-5r}{2}$

Substitute h into the expression for surface area.

$\boldsymbol{S = 2\pi rh + 2\pi r^2}$ ✓

$= 2\pi r\left(\frac{10-5r}{2}\right) + 2\pi r^2$

$= 10\pi r - 5\pi r^2 + 2\pi r^2$

$= \boldsymbol{10\pi r - 3\pi r^2}$ ✓

b **If the surface area has a local maximum, it occurs at a stationary point of S, that is, for r such that $\frac{dS}{dr} = 0$.** ✓

Differentiating $S(r)$ gives $\boldsymbol{\frac{dS}{dr} = 10\pi - 6\pi r}$ ✓, hence $\frac{dS}{dr} = 0$ for $r = \frac{5}{3}$ is a local maximum of S.

Note that the expression for S represents a downwards opening parabola, so any local maximum which occurs for $r > 0$ is also the global maximum of the function. This is confirmed by a graph of S.

Hence S has its maximum value for $\boldsymbol{r = \frac{5}{3}}$ **cm**.✓

Question 13 (4 marks)

(✓ = 1 mark)

a As points Y and Z are on the parabola $y = 9 - 3x^2$, the value of b is given by $b = 9 - 3a^2$. The area A of rectangle $XYZW$ is the product of the rectangle's width and height. It has width of $2a$ and height of b.

Hence $A = 2ab = 2a(9 - 3a^2) = \boldsymbol{18a - 6a^3}$.✓

b The local maximum occurs where $\frac{dA}{da} = 0$.

$\frac{dA}{da} = 18 - 18a^2 = 0$ when $\boldsymbol{a = \pm 1}$.✓

As $a > 0$, only $a = 1$ is a feasible solution.

A sign diagram for $\frac{dA}{da}$ shows the stationary point at **a = 1 is a local maximum.**✓

This maximum value is $A(1) = 18(1) - 6(1)^3 =$ **12 units²**.✓

Question 14 (4 marks)

(✓ = 1 mark)

Given the line $2x + y - 10 = 0$, $y = 10 - 2x$.

Therefore, any point P on the line is $(x, 10 - 2x)$.

The distance of point P from the origin is the length of line segment OP given by

$D(x) = \sqrt{(x-0)^2 + ((10-2x)-0)^2} = \mathbf{\sqrt{5x^2 - 40x + 100}}$. ✓

<u>Method 1</u>

The distance squared is

$D^2 = 5x^2 - 40x + 100 = 5(x^2 - 8x + 16) + 100 - 80 = 5(x-4)^2 + 20$.✓

D^2 will have a minimum value of 20 when $x = 4$.

When $x = 4$, $y = 10 - 2 \times 4 = 2$ so P is **(4, 2)**✓

The minimum distance is $\sqrt{20} = \mathbf{2\sqrt{5}}$ ✓

<u>Method 2</u>

Differentiating D^2 gives $\frac{dD^2}{dx} = \mathbf{10x - 40}$.✓

D^2 has a stationary point when $10x - 40 = 0 \rightarrow x = 4$

Note that the stationary points of D^2 are also stationary points of D, however, differentiation of D^2 is more straightforward.

A sign diagram shows that the stationary point at $x = 4$ is a local minimum.

Consideration of a graph of D^2 shows that this stationary point is also the global minimum of D^2 and hence D.

Hence, the distance OP is minimised when point $P(x, 10 - 2x)$ has $x = 4$, that is, P is **(4, 2).**✓

So the minimum length = $OP = \sqrt{4^2 + 2^2} = \sqrt{20} = \mathbf{2\sqrt{5}}$.✓

Question 15 (3 marks)

(✓ = 1 mark)

The area of the triangle $= \frac{1}{2}\textbf{base} \times \textbf{height} = \frac{1}{2}m\left(9 - m^2\right)$.✓

Set the derivative of the function equal to 0 and solve for *m*.

Substitute the positive value of *m* into the function to find the maximum area.

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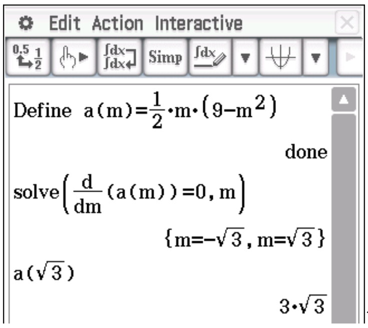

✓

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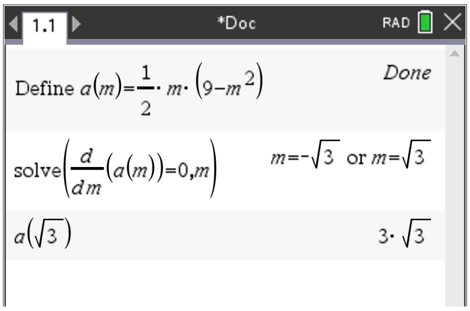

✓

The maximum area of the triangle is $3\sqrt{3}$ units².✓

Question 16 (13 marks)

(✓ = 1 mark)

a Point *G* is a local minimum of *f* (*x*). As such it occurs at an *x* value which is a solution to *f* ′(*x*) = 0. The graph shows that this *x* value is between 2 and 4.

Here $f'(x) = \frac{9}{64}x^2 - \frac{14}{32}x = \frac{x}{64}(9x - 28)$.✓

Hence *f* ′(*x*) = 0 has solutions at $x = 0, \frac{28}{9}$.

The stationary point at *x* = 0 is the local maximum of *f*, hence ***G* has *x*-coordinate $\frac{28}{9}$**.✓

Substitution of this value into *f*(*x*) gives the *y*-coordinate of *G*, which is $-\frac{50}{243}$.

Hence *G* is the point $\left(\frac{28}{9}, -\frac{50}{243}\right)$.✓

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b

The equation of the line through $A\left(0, \frac{1}{2}\right)$ and $B(4, 0)$ is

$y - y_1 = m(x - x_1)$

To find the slope, $m = \dfrac{0 - \frac{1}{2}}{4 - 0} = \mathbf{-\frac{1}{8}}$ ✓

The required equation is

$y - \frac{1}{2} = -\frac{1}{8}(x - 0)$

$\mathbf{y = -\frac{1}{8}x + \frac{1}{2}}$. ✓

c

The point $\left(x, -\frac{x}{8} + \frac{1}{2}\right)$ is on the line AB.

The point $\left(x, \frac{3x^3}{64} - \frac{7x^2}{32} + \frac{1}{2}\right)$ is on $f(x)$.

The distance between AB and $f(x)$ is

$d(x) = \left(-\frac{x}{8} + \frac{1}{2}\right) - \left(\frac{3x^3}{64} - \frac{7x^2}{32} + \frac{1}{2}\right) = -\frac{3x^3}{64} + \frac{7x^2}{32} - \frac{x}{8}$

$d(x)$ is a maximum when $d'(x) = 0$.

$\mathbf{-\frac{9x^2}{64} + \frac{14x}{32} - \frac{1}{8} = 0}$ ✓

$x = \dfrac{14 \pm 2\sqrt{31}}{9}$

The point that lies between $x = 2$ and $x = 4$ is $x = \dfrac{14 + 2\sqrt{31}}{9}$.

Also,

$d''(x) = -\frac{18x}{64} + \frac{14}{32}$

$\mathbf{d''\left(\frac{14 + 2\sqrt{31}}{9}\right) = -\frac{\sqrt{31}}{16} < 0}$ ✓

Therefore, the distance function is a maximum when $x = \dfrac{14 + 2\sqrt{31}}{9}$.

The x-coordinate of E is $x = \dfrac{14 + 2\sqrt{31}}{9}$ or $x = \dfrac{2\left(\sqrt{31} + 7\right)}{9}$. ✓

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d Given $v(x)=kx^{\frac{1}{2}}-mx^2$ and $v(4) = 0$, $\boldsymbol{v(4) = k(4)^{\frac{1}{2}} - m(4)^2 = 2k - 16m = 0}$.✓

Rearranging shows $\boldsymbol{k = 8m}$.✓

e Rewrite the speed expression as $v(x)=8mx^{\frac{1}{2}}-mx^2$. If the speed has a local maximum, it occurs when $\boldsymbol{v'(x)=4mx^{-\frac{1}{2}}-2mx=0}$ ✓, that is, for x such that $\frac{2m}{x^{\frac{1}{2}}}\left(2-x^{\frac{3}{2}}\right)=0$ or $x^{\frac{3}{2}}=2$.

This gives $\boldsymbol{x=2^{\frac{2}{3}}}$.✓

Recalling that m is positive, a sign diagram for $v'(x)$ shows that the speed of the train $v(x)$ has a local maximum at $\boldsymbol{x=2^{\frac{2}{3}}}$.✓

Cumulative examination: Calculator-free

Question 1 (3 marks)

(✓ = 1 mark)

For the curve $y = x^2(x + 1) = x^3 + x^2$, find $\frac{dy}{dx}$ at $x = –1$.

$\boldsymbol{\frac{dy}{dx} = 3x^2 + 2x}$ ✓

At $x = –1$, $\boldsymbol{\frac{dy}{dx} = 1}$ ✓

Use $y – y_1 = m(x – x_1)$ to find the equation of the tangent at the point (–1, 0).

$y – 0 = 1(x – (–1))$

So, $\boldsymbol{y = x + 1}$ is the equation of the tangent.✓

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Question 2 (6 marks)

(✓ = 1 mark)

a $f(x) = 2x^3 - 5x^2$.

$\boldsymbol{f'(x) = 6x^2 - 10x = 2x(3x-5)}$ ✓

$$f'\left(\frac{5}{3}\right) = 2\left(\frac{5}{3}\right)\left(3\left(\frac{5}{3}\right) - 5\right)$$

$$= 2\left(\frac{5}{3}\right)(0)$$

$$\boldsymbol{= 0}$$ ✓

$\boldsymbol{f''(x) = 12x - 10}$ ✓

$$f''\left(\frac{5}{3}\right) = 12\left(\frac{5}{3}\right) - 10$$

$$\boldsymbol{f''\left(\frac{5}{3}\right) = 10}$$ ✓

b $\boldsymbol{f''\left(\frac{5}{3}\right) > 0}$ ✓, so there is a **local minimum at** $\boldsymbol{x = \frac{5}{3}}$. ✓

Question 3 (4 marks)

(✓ = 1 mark)

$\boldsymbol{u = 2x^2 + 1 \Rightarrow u' = 4x}$ ✓, $\boldsymbol{v = \sqrt{x} = x^{\frac{1}{2}} \Rightarrow v' = \frac{1}{2}x^{-\frac{1}{2}}}$ ✓

$$\boldsymbol{\frac{vu' - uv'}{v^2} = \frac{x^{\frac{1}{2}}(4x) - (2x^2+1) \times \frac{1}{2}x^{-\frac{1}{2}}}{\left(x^{\frac{1}{2}}\right)^2}}$$ ✓

$\frac{vu' - uv'}{v^2} = \frac{8x^2 - (2x^2+1)}{2x^{\frac{3}{2}}}$, multiplying numerator and denominator by $2x^{\frac{1}{2}}$.

$$\boldsymbol{= \frac{6x^2 - 1}{2x^{\frac{3}{2}}}}$$ ✓

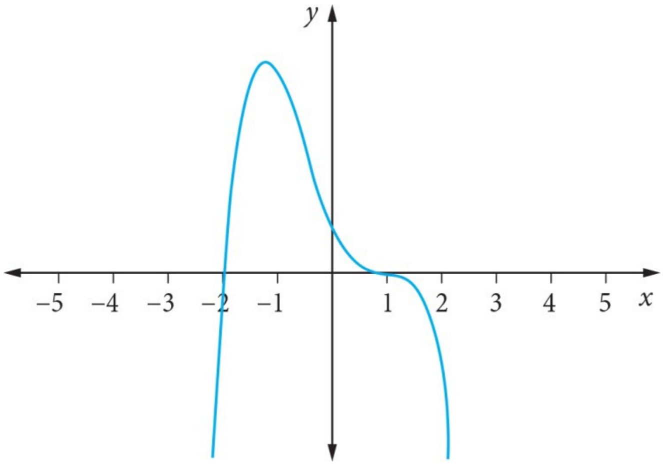
Question 4 (5 marks)

(✓ = 1 mark)

Since $f(-2) = 0$ and $f(1) = 0$, the co-ordinates of the x-intercepts are (–2, 0) and (1, 0).✓

$f'(1) = 0$ and $f''(1) = 0$ means a stationary point of inflection at (1, 0).✓

Since $x < -\frac{5}{4}$, $f'(x) > 0$ and $x < -\frac{5}{4} < x < 1$, $f'(x) < 0$, there is a stationary point (local maximum) at $x = -\frac{5}{4}$.✓



correct general shape✓

correct x-intercepts shown ✓

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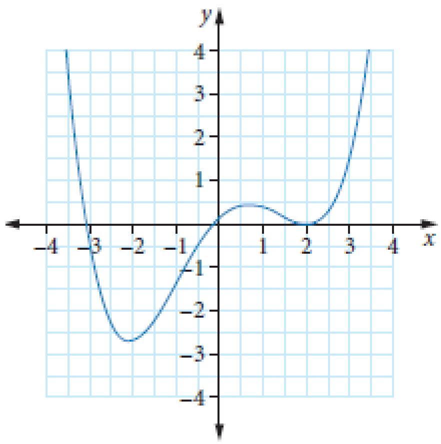
Question 5 (5 marks)

(✓ = 1 mark)

There is a stationary point of inflection at $x = 2$, so $f'(2) = 0$.✓

There is a local maximum turning point at approximately $x = -3$, $f'(x) > 0$ for $x < -3$ and $f'(x) > 0$ in $-3 < x < 0$. Hence, $f'(-3) = 0$.✓

There is a local minimum turning point at $x = 0$, $f'(x) > 0$ for $0 < x < 2$ and $f'(x) > 0$ for $x > 2$. Hence $f'(0) = 0$.✓



correct general shape✓

correct x-intercepts shown ✓

Cumulative examination 2: Calculator-assumed

Question 1 [SCSA MM2016 Q11] (3 marks)

(✓ = 1 mark)

$a = b = l, \quad C = \frac{\pi}{3}$

$A = \frac{1}{2} l^2 \sin\frac{\pi}{3} = \frac{\sqrt{3}}{4} l^2$

$l = 10, \quad \delta l = 0.1$

$\delta A \approx \frac{dA}{dl}\delta l = \frac{2\sqrt{3}}{4}(10)0.1$

$\delta A = 0.87$

Approximate change in area of 0.87 cm^2

sets up an equation for area in terms of one variable✓

uses increments formula with correct parameters✓

determines approximate change in area✓

Question 2 [SCSA MM2017 Q17] (6 marks)

(✓ = 1 mark)

a $375 = \pi x^2 h$

$\therefore h = \frac{375}{\pi x^2}$

$S = 2\pi x^2 + 2\pi x h$

$= 2\pi x^2 + 2\pi x\left(\frac{375}{\pi x^2}\right)$

$= 2\pi x^2 + \frac{750}{x}$

uses volume formula to determine *h* in terms of *x*✓

demonstrates substitution of *h* into surface area formula and simplifies to show result✓

b

$$S = 2\pi x^2 + \frac{750}{x}$$
$$\frac{dS}{dx} = 4\pi x - \frac{750}{x^2}$$
$$0 = 4\pi x - \frac{750}{x^2}$$
$$x = 3.908 \text{ cm}$$
$$\frac{d^2S}{dx^2} = 4\pi + \frac{1500}{x^3}$$
$$\left.\frac{d^2S}{dx^2}\right|_{x=3.908} = +ve(37.7) \Rightarrow \text{Min}$$

When $x = 3.908$, $h = 7.816$

Cans have a radius of 3.9 cm and a height of 7.8 cm to minimise surface area.

determines first derivate✓

equates to zero to find *x* ✓

justifies minimum with second derivative or other suitable method✓

states dimension of the can✓

Question 3 [SCSA MM2018 Q15] (5 marks)

(✓ = 1 mark)

$$\text{For } HPI: \begin{cases} P'(6) = 0 \\ P''(6) = 0 \end{cases}$$

$$P'(t) = 3t^2 + 2at + b$$
$$P''(t) = 6t + 2a$$
$$0 = 108 + 12a + b$$
$$0 = 36 + 2a$$

Solving gives:

$a = -18$

$b = 108$

determines first derivative✓

determines second derivative✓

equates first and second derivatives to zero when $t = 6$✓

determines the value of *a*✓

determines the value of *b*✓

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Question 4 (10 marks)

(✓ = 1 mark)

a $s(2)=2^3-4(2)^2+4(2)-10=-10$ ✓

$s(0)=0^3-4(0)^2+4(0)-10=-10$

$\frac{s(2)-s(0)}{2-0}=\frac{-10-(-10)}{2}=0$ ✓

The change in displacement is 0 metres.

b $s'(t)=3t^2-8t+4$

$s'(5)=3(5)^2-8(5)+4$ ✓

$=39$

The velocity after 5 seconds is 39 m/s.✓

c $s'(t)=0$

$3t^2-8t+4=0$ ✓

$(3t-2)(t-2)=0$

$t=\frac{2}{3}, t=2$ ✓

The particle is at rest at $t=\frac{2}{3}$ and $t=2$ seconds.

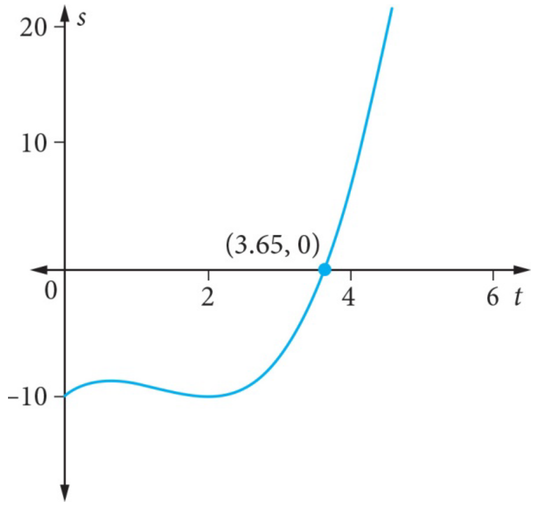
d $s''(t)=6t-8$ ✓

$s''(0)=6(0)-8=-8$ ✓

The initial acceleration is –8 m/s^2.

e $s(t)=0$

$t^3-4t^2+4t-10=0 \Rightarrow t=3.654$



Turning point at $t=\frac{2}{3}$, so $s\left(\frac{2}{3}\right)=-8.815$.

Initial displacement is $s(0)=-10$, hence distance in $0\le t\le\frac{2}{3}$ is 10 – 8.815 = **1.185 m**✓

Another turning point at $t=2$, so $s(2)=-10$.

Distance in $\frac{2}{3}\le t\le 2$ is 10 – 8.815 = 1.185 m

Total distance is 1.185 + 1.185 = **2.37 m**✓

Question 5 (6 marks)

(✓ = 1 mark)

a Let h be the height of the tank.

$V = 8$

$xyh = 8$

$$h = \frac{8}{xy} \checkmark$$

$$A = xy + 2yh + 2xh$$

$$= xy + 2y \times \frac{8}{xy} + 2x \times \frac{8}{xy} \checkmark$$

$$A = xy + \frac{16}{x} + \frac{16}{y}$$

b

$$\frac{dA}{dy} = x - \frac{16}{y^2} \checkmark$$

For maximum, $\frac{dA}{dy} = 0$

$$x - \frac{16}{y^2} = 0 \Rightarrow y = \frac{4}{\sqrt{x}}$$, take positive root since $y > 0$.✓

Hence $A = x \times \frac{4}{\sqrt{x}} + \frac{16}{x} + \frac{16}{\frac{4}{\sqrt{x}}} \checkmark$

$$= 8\sqrt{x} + \frac{16}{x}$$

So, $a = 8$ and $b = 16$. ✓

Chapter 3 – Integrals

EXERCISE 3.1 The anti-derivative

Question 1

Write the function as a derivative.

$$\frac{dy}{dx} = x^3 + 3x^2 - 4x$$

Integrate each term using the formula $\int ax^n dx = \frac{ax^{n+1}}{a(n+1)} + c.$

$$\int \left(x^3 + 3x^2 - 4x\right) dx = \frac{x^4}{4} + 3\frac{x^3}{3} - 4\frac{x^2}{2} + c$$

$$\boldsymbol{y = \frac{x^4}{4} + x^3 - 2x^2 + c.}$$

Question 2

Integrate using $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c.$

$$\int (4x-1)^3 \; dx = \frac{1}{4(3+1)}(4x-1)^{3+1} + c$$

$$\boldsymbol{= \frac{1}{16}(4x-1)^4 + c.}$$

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Question 3

a Write the function as an integral, integrate each term using $\int ax^n dx = \frac{ax^{n+1}}{a(n+1)}$ and add the constant of integration.

$$\int (x^2 - 3x + 2)\,dx = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + c = \frac{x^3}{3} - \frac{3x^2}{2} + 2x + c.$$

b Expand and simplify brackets.

$$(x-3)(2x+4) = 2x^2 - 2x - 12$$

Integrate each term using $\int ax^n dx = \frac{ax^{n+1}}{a(n+1)}$ and add the constant of integration.

$$\int (2x^2 - 2x - 12)\,dx = \frac{2}{3}x^3 - x^2 - 12x + c = \frac{2x^3}{3} - x^2 - 12x + c.$$

c Simplify the function.

$$\frac{x^2 - 2x}{x} = \frac{x(x-2)}{x} = x - 2$$

Integrate each term using $\int ax^n dx = \frac{ax^{n+1}}{a(n+1)}$ and add the constant of integration.

$$\int (x-2)\,dx = \frac{1}{2}x^2 - 2x + c = \frac{x^2}{2} - 2x + c.$$

d Write each term in the form ax^n.

$$\sqrt{x}-\frac{1}{x^2}-3=x^{\frac{1}{2}}-x^{-2}-3$$

Integrate each term using $\int ax^n dx=\frac{ax^{n+1}}{a(n+1)}$ and add the constant of integration.

$$\int\left(x^{\frac{1}{2}}-x^{-2}-3\right)dx=\frac{2}{3}x^{\frac{3}{2}}-\frac{x^{-1}}{-1}-3x+c.$$

$$=\frac{2}{3}x^{\frac{3}{2}}+\frac{1}{x}-3x+c=\frac{2x^{\frac{3}{2}}}{3}+\frac{1}{x}-3x+c$$

e $\sqrt{2x-3}=(2x-3)^{\frac{1}{2}}$

Integrate using $\int(ax+b)^n\,dx=\frac{(ax+b)^{n+1}}{a(n+1)}+c$

$$\int(2x-3)^{\frac{1}{2}}\,dx=\frac{(2x-3)^{\frac{3}{2}}}{2\times\frac{3}{2}}+c.$$

$$=\frac{1}{3}(2x-3)^{\frac{3}{2}}+c=\frac{(2x-3)^{\frac{3}{2}}}{3}+c$$

f Write each term in the form ax^n.

$$\sqrt{x}\left(x^2-2x+3\right)=x^{\frac{5}{2}}-2x^{\frac{3}{2}}+3x^{\frac{1}{2}}$$

Integrate each term using $\int ax^n dx=\frac{ax^{n+1}}{a(n+1)}$ and add the constant of integration.

$$\int\left(x^{\frac{5}{2}}-2x^{\frac{3}{2}}+3x^{\frac{1}{2}}\right)dx=\frac{2}{7}x^{\frac{7}{2}}-\frac{4}{5}x^{\frac{5}{2}}+2x^{\frac{3}{2}}+c\,.$$

$$\sqrt{x}\left(x^2-2x+3\right)=\frac{2}{7}x^{\frac{7}{2}}-\frac{4}{5}x^{\frac{5}{2}}+2x^{\frac{3}{2}}+c=\frac{2x^{\frac{7}{2}}}{7}-\frac{4x^{\frac{5}{2}}}{5}+2x^{\frac{3}{2}}+c$$

g $\dfrac{1}{(2x-3)^2}=(2x-3)^{-2}$

Integrate using $\int (ax+b)^n\,dx=\dfrac{(ax+b)^{n+1}}{a(n+1)}+c$

$$\int (2x-3)^{-2}\,dx=\frac{(2x-3)^{-1}}{2\times -1}+c$$

$$=-\frac{1}{2(2x-3)}+c$$

$$\boldsymbol{\int \frac{1}{(2x-3)^2}dx=-\frac{1}{2(2x-3)}+c}$$

h $\sqrt[3]{3x-4}=(3x-4)^{\frac{1}{3}}$

Integrate using $\int (ax+b)^n\,dx=\dfrac{(ax+b)^{n+1}}{a(n+1)}+c$

$$\int (3x-4)^{\frac{1}{3}}\,dx=\frac{(3x-4)^{\frac{4}{3}}}{3\times\frac{4}{3}}+c$$

$$=\frac{1}{4}(3x-4)^{\frac{4}{3}}+c$$

$$\boldsymbol{\int \sqrt[3]{3x-4}dx=\frac{1}{4}(3x-4)^{\frac{4}{3}}+c=\frac{(3x-4)^{\frac{4}{3}}}{4}+c}$$

Question 4

$f'(x) = 2x$

Integrate each term and simplify.

$$f(x) = \frac{2x^2}{2} + c = x^2 + c$$

Find the value of c by substituting $x = 2$, $y = 1$.

When $x = 2$ and $y = 1$,

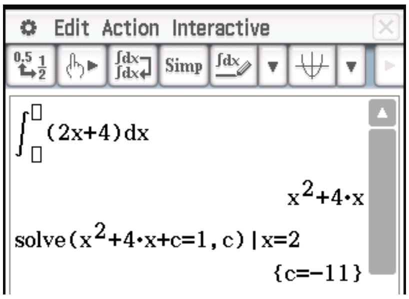
$$f(2) = 1 = (2)^2 + c$$
$$1 - 4 = c$$
$$\therefore c = -3$$

State the answer including the value of c.

$$\boldsymbol{f(x) = x^2 - 3}$$

Question 5

ClassPad



1 Highlight the expression then tap

Interactive > Calculation > ∫.

2 Add + c to the expression as it is not included in the solution.

3 Set the expression including the +c equal to 0 and solve by including the condition that $x = 2$.

4 State the answer.

$$y = x^2 + 4x - 11$$

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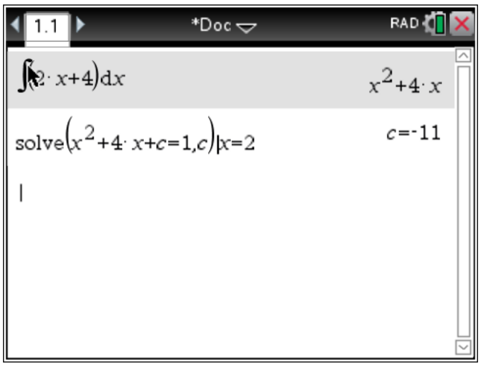

1 Press menu > Calculus > Integral.

2 Enter the derivative expression followed by dx.

3 Add +c to the expression as it is not included in the solution.

4 Set the expression including the +c equal to 0 and solve by including the condition that $x = 2$.

5 State the answer.

$$y = x^2 + 4x - 11$$

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Question 6

$$\frac{4}{\sqrt[3]{4-2x}}=4(4-2x)^{-\frac{1}{3}}$$

Integrate using $\int(ax+b)^n\,dx=\frac{(ax+b)^{n+1}}{a(n+1)}+c$

$$\int 4(4-2x)^{-\frac{1}{3}}\,dx=\frac{4(4-2x)^{\frac{2}{3}}}{-2\times\frac{2}{3}}$$

$$=-3(4-2x)^{\frac{2}{3}}$$

$$f(x)=-3(4-2x)^{\frac{2}{3}}+c$$

Find c using the fact that $x=-2$, $y=10$.

$$-3(4-2\times(-2))^{\frac{2}{3}}+c=10$$
$$-3(2^3)^{\frac{2}{3}}+c=10$$
$$-12+c=10$$
$$c=22$$

So $\boldsymbol{f(x)=-3(4-2x)^{\frac{2}{3}}+22=-3(-2x+4)^{\frac{2}{3}}+22}$

Question 7 (2 marks)

(✓ = 1 mark)

The anti-derivative of the expression $3x^2+4x^3-2$ is

$$\int(3x^2+4x^3-2)\,dx=\boldsymbol{\frac{3x^3}{3}+\frac{4x^4}{4}-2x+c}$$ ✓(correct terms)

$$=\boldsymbol{x^3+x^4-2x+c}$$ ✓ (include constant of integration)

Question 8 (2 marks)

(✓ = 1 mark)

$$\int \frac{1}{(3x+4)^4}dx = \int (3x+4)^{-4}dx \checkmark$$

$$= \frac{(3x+4)^{-4+1}}{3(-4+1)} = \frac{(3x+4)^{-3}}{-9} = \frac{-1}{9(3x+4)^3} + c \checkmark$$

Question 9 (2 marks)

(✓ = 1 mark)

Integrate using $\int (ax+b)^n\,dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + c$

$$\int (4-2x)^{-5}\,dx = \frac{(4-2x)^{-4}}{(-2)\times(-4)} \checkmark$$

$$= \frac{1}{8}(4-2x)^{-4} + c \text{ or } \frac{1}{8(4-2x)^4} + c \checkmark$$

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Question 10 (2 marks)

(✓ = 1 mark)

$f'(x) = 3x^2 - 2x$

$f(x) = \frac{3x^3}{3} - \frac{2x^2}{2} + c$

$\mathbf{= x^3 - x^2 + c}$ ✓

$f(4) = 4^3 - 4^2 + c = 0$

$c = -64 + 16$

$c = -48$

$\boldsymbol{f(x) = x^3 - x^2 - 48}$ ✓

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Question 11 (2 marks)

(✓ = 1 mark)

$f'(x) = ax^2 - bx = x(ax - b)$

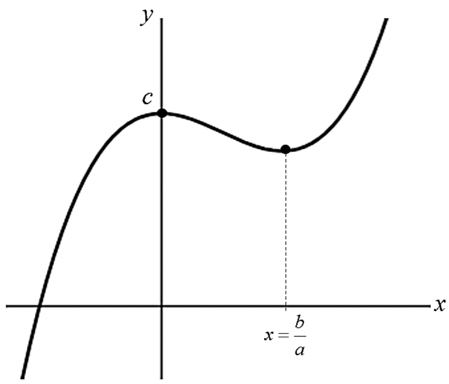
$f'(x) = 0 \Rightarrow x = 0,\ x = \frac{b}{a}$

$f(x) = \frac{1}{3}ax^3 - \frac{1}{2}bx^2 + c$, whose graph is a cubic.

For $x < 0$, $f'(x) > 0$, and for $0 < x < \frac{b}{a}$, $f'(x) < 0$.

Hence $f(x)$ has a local maximum at $x = 0$.

For $x > \frac{b}{a}$, $f'(x) > 0$. Hence $f(x)$ has a local minimum at $x = \frac{b}{a}$.



Answer can vary. This is one possible answer. Check with your teacher for other possible answers.

correct reasoning about the general shape✓

graph correctly drawn, showing turning points✓

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Question 12 (2 marks)

(✓ = 1 mark)

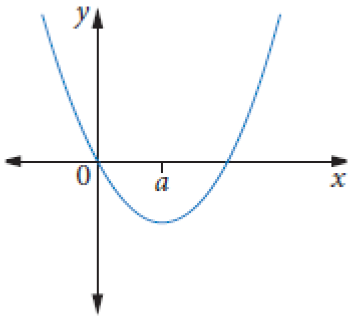
a Look at the *sign* of the derivative graph. As the gradient is negative, then zero, then positive, it reaches a minimum point corresponding to the x-intercept.

The graph of the anti-derivative is a parabola.

$f'(x)=0$ at $x=a$, $f'(x)<0$ for $x<a$, $f'(x)>0$ for $x>a$

Hence the parabola has a minimum turning point at $x=a$.

Since $a>0$, the co-ordinates of the turning point will be in the 1st or in the 4 th quadrant.



✓

Answer can vary. This is one possible answer. Check **with your teacher for other possible answers.**

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b Look at the *sign* of the derivative graph. It starts off positive (above the x-axis), which means that the function is increasing. It becomes zero, then negative, so the function reaches a maximum point, then is decreasing. It becomes zero again, then positive, so the function reaches a minimum point, then is increasing again.

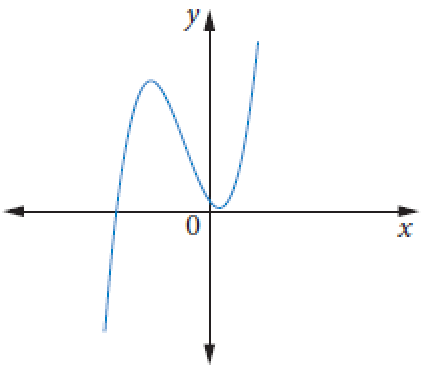
or

The graph of the derivative has two x-intercepts. These are the co-ordinates of the stationary points of the anti-derivative, which is a cubic.

To the left of the x-intercept that lies on the negative x-axis, $g'(x)$ is positive, and to the right, $g'(x)$ is negative. This means the x-intercept represents a local maximum on the graph of $g(x)$.

To the left of the x-intercept that lies on the positive x-axis, $g'(x)$ is negative, and to the right, $g'(x)$ is positive. This means the x-intercept represents a local minimum on the graph of $g(x)$.

The x co-ordinate of the turning point of $g'(x)$ is negative and is the x co-ordinate of the point of inflection on the graph of $g(x)$.



✓

Question 13 (2 marks)

(✓ = 1 mark)

$$\int \frac{2x-3}{\sqrt{x^2-3x}}dx = \int \frac{1}{\sqrt{u}}du = \int u^{-\frac{1}{2}}du\text{, where } u = x^2 - 3x \checkmark$$

$$= 2u^{\frac{1}{2}}$$

$$= 2\sqrt{x^2-3x} + c \checkmark$$

Question 14 (2 marks)

(✓ = 1 mark)

$$f'(x) = 2x^2 - \frac{1}{4}x^{-\frac{2}{3}}$$

$$f(x) = \frac{2}{3}x^3 - \frac{\frac{1}{4}}{\frac{1}{3}}x^{\frac{1}{3}} + c$$

$$= \frac{2}{3}x^3 - \frac{3}{4}x^{\frac{1}{3}} + c \checkmark$$

$$f(1) = \frac{2}{3} - \frac{3}{4} + c = -\frac{7}{4}$$

$$c = -\frac{2}{3} + \frac{3}{4} - \frac{7}{4}$$

$$= -\frac{8}{12} + \frac{9}{12} - \frac{21}{12}$$

$$= -\frac{20}{12}$$

$$= -\frac{5}{3}$$

$$f(x) = \frac{2}{3}x^3 - \frac{3}{4}x^{\frac{1}{3}} - \frac{5}{3} \checkmark$$

Question 15 (2 marks)

(✓ = 1 mark)

The anti-derivative of $\frac{1}{(2x-1)^3}$ is

$$\int \frac{1}{(2x-1)^3}\,dx = \int (2x-1)^{-3}\,dx$$

$$= \frac{1}{2(-3+1)}(2x-1)^{-3+1} + c \checkmark$$

$$= \frac{(2x-1)^{-2}}{-4} + c$$

$$= -\frac{1}{4(2x-1)^2} + c \checkmark$$

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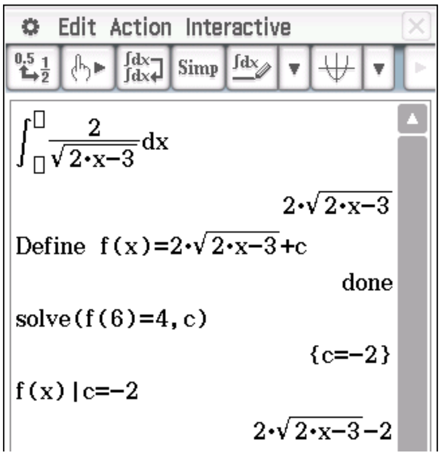
Question 16 (2 marks)

(✓ = 1 mark)

Find the anti-derivative if $f'(x)$. Define the function $f(x)$ by including the $+c$.

Solve $f(6)=4$ to determine the value of c. Substitute the c value into the function.

ClassPad



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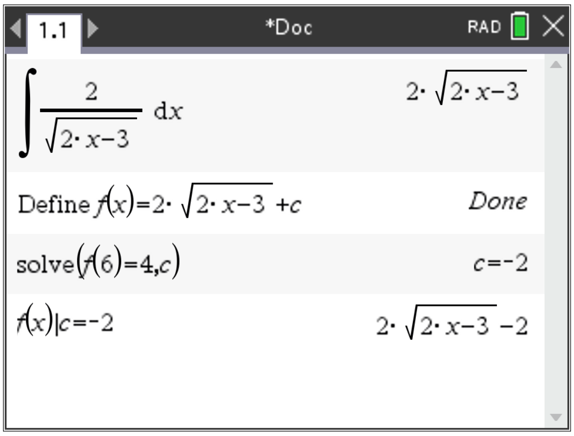

Correct value of *c*. ✓

Correct answer✓

Question 17 (2 marks)

(✓ = 1 mark)

$$f'(x)=g'(x)+3$$
$$f(x)=g(x)+3x+c$$

For $x=0$,

$$f(0)=g(0)+3\times 0+c$$
$$2=1+3\times 0+c$$

$\boldsymbol{c=1}$✓

$\boldsymbol{f(x)=g(x)+3x+1}$✓

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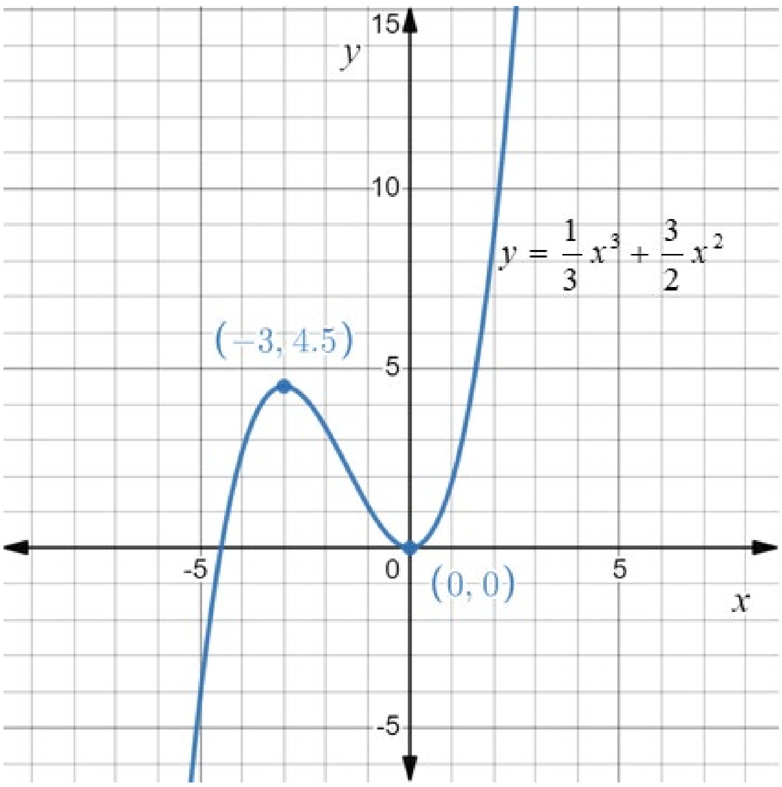
EXERCISE 3.2 Approximating areas under curves

Question 1

Consider the graph of $y = x^2 + 3x$ (parabola) and its anti-derivative $y = \frac{x^3}{3} + \frac{3x^2}{2} + c$ (cubic), with $c = 0$. So, the parabola is the derivative of the cubic. The x-intercepts of the parabola are at the same x values as the turning points on the cubic graph, i.e., $x = 0$ and -3.



Question 2

$$f'(x) = 2x - x^{\frac{2}{3}}$$

$$f(x) = 2\left[\frac{x^{1+1}}{1+1}\right] - \left[\frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1}\right] + c$$

$$= x^2 - \frac{3}{5}x^{\frac{5}{3}} + c$$

Given $f(1) = -1$.

So $-1 = (1)^2 - \frac{3}{5}(1)^{\frac{5}{3}} + c$

$c = -1 - 1 + \frac{3}{5} = -\frac{7}{5}$

$$f(x) = x^2 - \frac{3}{5}x^{\frac{5}{3}} - \frac{7}{5}$$

Question 3

a Find the height of each rectangle.

$f(1) = 1^2 + 2 = 3$

$f(1.5) = (1.5)^2 + 2 = 4.25$

Find the area of each rectangle.

$A_1 = 0.5 \times 3 = 1.5$

$A_2 = 0.5 \times 4.25 = 2.125$

The total area is $A = A_1 + A_2 =$ **3.625 units²**.

b Find the height of each rectangle.

$f(1) = 1^3 = 1$

$f(2) = 2^3 = 8$

$f(3) = 3^3 = 27$

$f(4) = 4^3 = 64$

Find the area of each rectangle.

$A_1 = 1 \times 1 = 1$

$A_2 = 1 \times 8 = 8$

$A_3 = 1 \times 27 = 27$

$A_4 = 1 \times 64 = 64$

The total area is $A = A_1 + A_2 + A_3 + A_4 =$ **100 units²**.

c Find the height of each rectangle.

$f(1) = 4(1) - 1^2 = 3$

$f(2) = 4(2) - 2^2 = 4$

$f(3) = 4(3) - 3^2 = 3$

Find the area of each rectangle.

$A_1 = 1 \times 3 = 3$

$A_2 = 1 \times 4 = 4$

$A_3 = 1 \times 3 = 3$

The total area is $A = A_1 + A_2 + A_3 =$ **10 units²**.

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d Find the height of each rectangle.

$f(3) = \sqrt{3+1} = 2$

$f(4) = \sqrt{4+1} = \sqrt{5} = 2.236\ldots$

$f(5) = \sqrt{5+1} = \sqrt{6} = 2.449\ldots$

$f(6) = \sqrt{6+1} = \sqrt{7} = 2.645\ldots$

Find the area of each rectangle.

$A_1 = 1 \times 2 = 2$

$A_2 = 1 \times 2.236\ldots = 2.236\ldots$

$A_3 = 1 \times 2.449\ldots = 2.449\ldots$

$A_4 = 1 \times 2.645\ldots = 2.645\ldots$

The total area is $A = A_1 + A_2 + A_3 + A_4 \approx$ **9.33 units^2**.

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Question 4

a To find the area of $y = x^2$, between $x = 0$ and $x = 1$, first find the height of each rectangle.

$$f\left(\frac{1}{5}\right) = \frac{1}{25}$$
$$f\left(\frac{2}{5}\right) = \frac{4}{25}$$
$$f\left(\frac{3}{5}\right) = \frac{9}{25}$$
$$f\left(\frac{4}{5}\right) = \frac{16}{25}$$
$$f(1) = 1$$

Find the area of each rectangle.

$$A_1 = \frac{1}{5} \times \frac{1}{25} = \frac{1}{125}$$

$$A_2 = \frac{1}{5} \times \frac{4}{25} = \frac{4}{125}$$

$$A_3 = \frac{1}{5} \times \frac{9}{25} = \frac{9}{125}$$

$$A_4 = \frac{1}{5} \times \frac{16}{25} = \frac{16}{125}$$

$$A_5 = \frac{1}{5} \times 1 = \frac{1}{5}$$

The total area is $A = A_1 + A_2 + A_3 + A_4 + A_5 = \frac{55}{125} =$ **0.44 units²**.

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b To find the area of $y = 2^x + 3$, between $x = 0$ and $x = 3$, first find the height of each rectangle.

$$f\left(\frac{1}{2}\right) = 2^{\frac{1}{2}} + 3 = 4.414...$$

$$f(1) = 2^1 + 3 = 5$$

$$f\left(\frac{3}{2}\right) = 2^{\frac{3}{2}} + 3 = 5.828...$$

$$f(2) = 2^2 + 3 = 7$$

$$f\left(\frac{5}{2}\right) = 2^{\frac{5}{2}} + 3 = 8.656....$$

$$f(3) = 2^3 + 3 = 11$$

Find the area of each rectangle.

$$A_1 = \frac{1}{2} \times 4.414... = 2.207...$$

$$A_2 = \frac{1}{2} \times 5 = 2.5$$

$$A_3 = \frac{1}{2} \times 5.828... = 2.914...$$

$$A_4 = \frac{1}{2} \times 7 = 3.5$$

$$A_5 = \frac{1}{2} \times 8.656... = 4.328...$$

$$A_6 = \frac{1}{2} \times 11 = 5.5$$

The total area is $A = A_1 + A_2 + A_3 + A_4 + A_5 + A_6 \approx$ **20.95 units²**.

c To find the area of $f(x)=\frac{2}{x+1}$, between $x = 1$ and $x = 4$, first find the height of each rectangle.

$f(2)=\frac{2}{2+1}=\frac{2}{3}$

$f(3)=\frac{2}{3+1}=\frac{1}{2}$

$f(4)=\frac{2}{4+1}=\frac{2}{5}$

Find the area of each rectangle.

$A_1=1\times\frac{2}{3}=\frac{2}{3}$

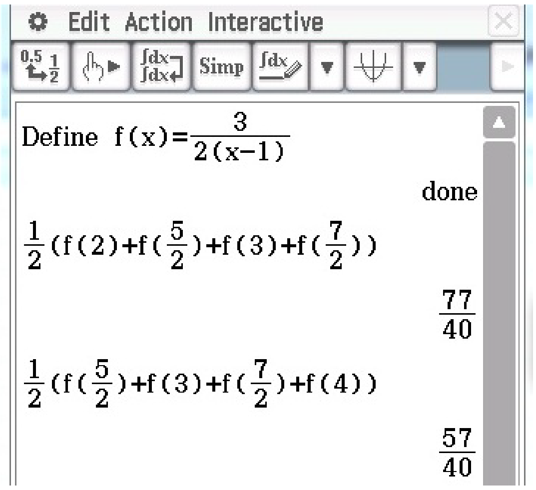
$A_2=1\times\frac{1}{2}=\frac{1}{2}$

$A_3=1\times\frac{2}{5}=\frac{2}{5}$

The total area is $A = A_1 + A_2 + A_3 = \frac{47}{30} \approx$ **1.57 units^2**.

Question 5

ClassPad



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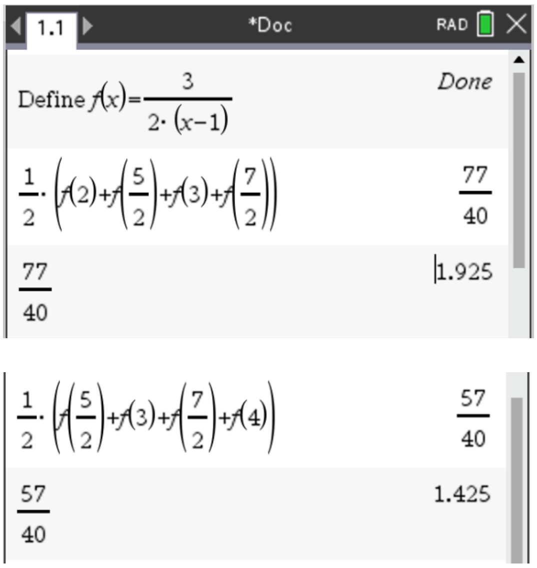

ClassPad:

1 Define **f(x)** as shown above.

2 Calculate the sum of the rectangle using the left-hand values of f(x).

3 Calculate the sum of the rectangles using the right-hand values of f(x).

TI-Nspire:

1 Define **f(x)** as shown above.

2 Calculate the sum of the 4 rectangles using the left-hand values of f(x).

3 Calculate the sum of the 4 rectangles using the right-hand values of f(x).

The approximations for the area under the graph are, respectively, closest to 1.93 and 1.43.

Take average. The area under the graph is $\frac{1}{2}(1.93 + 1.43) = 1.68$

Question 6 (3 marks)

(✓ = 1 mark)

To find the area of $y = \cos(x)$ between $x = 0$ and $x = \frac{\pi}{2}$, first find the height of each rectangle.

$$f(0) = \cos(0) = 1$$
$$f\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \quad ✓$$
$$f\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

Find the area of each rectangle.

$$A_1 = \frac{\pi}{6} \times 1 = \frac{\pi}{6}$$
$$A_2 = \frac{\pi}{6} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}\pi}{12} \quad ✓$$
$$A_3 = \frac{\pi}{6} \times \frac{1}{2} = \frac{\pi}{12}$$

The total area is $A = A_1 + A_2 + A_3 = \frac{\pi}{6}\left(1 + \frac{\sqrt{3}}{2} + \frac{1}{2}\right) = \frac{\pi}{12}\left(\sqrt{3} + 3\right)$. ✓

Question 7 (4 marks)

(✓ = 1 mark)

a $A=\frac{1}{2}[(a+b)h]$

$A=\frac{1}{2}\left[\left((f(0)+f(1))\times 1\right)+\left((f(1)+f(2))\times 1\right)+\left((f(2)+f(3))\times 1\right)\right]$

$\mathbf{A=\frac{1}{2}[9+8+8+5+5+0]}$ ✓

$=\frac{35}{2}$

$=$ **17.5 units**^{2}✓

b $A=\frac{1}{2}[(a+b)h]$

$A=\frac{1}{2}\left[\left(\left(f(0)+f\left(\frac{1}{2}\right)\right)\times\frac{1}{2}\right)+\left(\left(f\left(\frac{1}{2}\right)+f(1)\right)\times\frac{1}{2}\right)+\left(\left(f(1)+f\left(\frac{3}{2}\right)\right)\times\frac{1}{2}\right)+\left(\left(f\left(\frac{3}{2}\right)+f(2)\right)\times\frac{1}{2}\right)\right]$

$\mathbf{A=\frac{1}{2}\times\frac{1}{2}[(9+8.75)+(8.75+8)+(8+6.75)+(6.75+5)]}$ ✓

$=\frac{61}{4}$

= 15.25 units^2✓

Question 8 (2 marks)

(✓ = 1 mark)

We know that the more rectangles we use, the smaller the width of each rectangle and the closer we get to the actual area. ✓ **Therefore, the approximation will be most accurate if the value of *n* is larger and the value of *h* is smaller.**✓

Question 9 [SCSA MM2017 Q9] (8 marks)

(✓ = 1 mark)

a lower limit = 20 × 0.5 + 21 × 0.5 + 24 × 0.5

= 10 + 10.5 + 12

= 32.5

upper limit = 21 × 0.5 + 24 × 0.5 + 29 × 0.5

= 10.5 + 12 + 14.5

= 37

therefore $\int_0^{1.5} f(x)\,dx$ is between these values as this is the area under the curve

shows a calculation to produce an underestimate of area✓

shows a calculation to produce an overestimate of area ✓

explains the limits in terms of area✓

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b New under and over are:

under est = $24 \times 0.5 + 29 \times 0.5 + 36 \times 0.5 + 45 \times 0.5$

$= 67$

over est = $29 \times 0.5 + 36 \times 0.5 + 45 \times 0.5 + 56 \times 0.5$

$= 83$

$$\text{area} \approx \frac{67+83}{2}$$

$= 75$

determines the underestimate✓

determines the overestimate✓

finds the mean to produce best estimate of area✓

c
- by reducing the width of the rectangles and, therefore, using more rectangles to estimate the area the error in the estimate would be reduced
- determining the function and using calculus

states one reason✓

states two reasons✓

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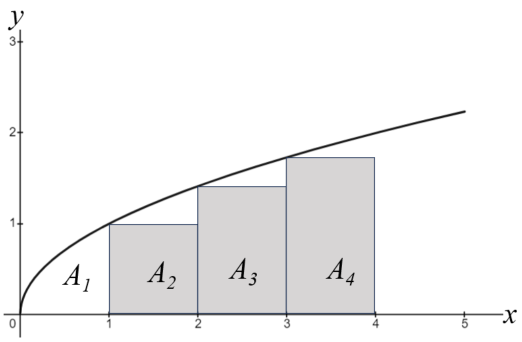
Question 10 (3 marks)

(✓ = 1 mark)



Underestimate

$y = f(x) = \sqrt{x}$

Find the height of each rectangle.

$f(0) = 0$
$f(1) = 1$
$f(2) = 1.414$
$f(3) = 1.732$

Find the area of each rectangle.

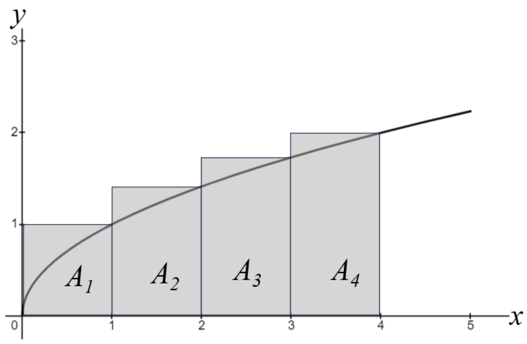
$A_1 = 1 \times 0 = 0$

$A_2 = 1 \times 1 = 1$

$A_3 = 1 \times 1.414 = 1.414$

$A_4 = 1 \times 1.732 = 1.732$

The total area is **4.146** units^2. ✓



Overestimate

$y = f(x) = \sqrt{x}$

Find the height of each rectangle.

$f(1) = 1$
$f(2) = 1.414$
$f(3) = 1.732$
$f(4) = 2$

Find the area of each rectangle.

$A_1 = 1 \times 1 = 1$

$A_2 = 1 \times 1.414 = 1.414$

$A_3 = 1 \times 1.732 = 1.732$

$A_4 = 1 \times 2 = 2$

The total area is **6.146** units^2. ✓

The average of the two areas is $\dfrac{4.146 + 6.146}{2} = 5.146$

The estimated area is 5.146 units^2. ✓

Question 11 (2 marks)

(✓ = 1 mark)

No.✓ The width of each rectangle is 1 and there are 4 rectangles, so the area would be

$A = f(1) + f(2) + f(3)$.✓

Question 12 (3 marks)

(✓ = 1 mark)

To find the area of $y = 10 - x^2$, between $x = 0$ and $x = 2$, first find the height of each rectangle.

$$f\left(\frac{1}{2}\right) = 10 - \frac{1}{4} = \frac{39}{4} = 9.75$$
$$f(1) = 10 - 1^2 = 9$$
$$f\left(\frac{3}{2}\right) = 10 - \frac{9}{4} = \frac{31}{4} = 7.75$$
$$f(2) = 10 - 4 = 6$$

✓

Find the area of each rectangle.

$$A_1 = \frac{1}{2} \times 9.75 = 4.875$$
$$A_2 = \frac{1}{2} \times 9 = 4.5$$
$$A_3 = \frac{1}{2} \times 7.75 = 3.875$$
$$A_4 = \frac{1}{2} \times 6 = 3$$

✓

$A = 4.875 + 4.5 + 3.875 + 3 =$ **16.25**✓

EXERCISE 3.3 The definite integral and the fundamental theorem of calculus

Question 1

$f(x) = x^2 + 1$

$f(0) = 1,\ f(0.5) = 1.25,\ f(1) = 2,\ f(1.5) = 3.25,\ f(2) = 5$

estimate below $= 0.5\ (1 + 1.25 + 2 + 3.25) = 3.75$

estimate above $= 0.5\ (1.25 + 2 + 3.25 + 5) = 5.75$

approximate area $= 0.5\ (3.75 + 5.75) = 4.75$

The area under the curve $y = x^2 + 1$ from $x = 0$ to $x = 2$ is **4.75 units²**.

Question 2

By reducing the width of the rectangles and, therefore, using more rectangles to estimate the area, the error in the estimate would be reduced. This is because there will be fewer gaps between the rectangles and the curve of the function.

Question 3

a Integrate $4x$.

$$\int_1^3 4x\,dx = \left[\frac{4x^2}{2}\right]_1^3$$

Substitute the limits of the integral $x = 3$ and $x = 1$ and subtract: $F(b) - F(a)$.

$$\left[2x^2\right]_1^3 = 2(3)^2 - 2(1)^2 = \mathbf{16}$$

b Integrate $7x^6$.

$$\int_0^2 7x^6 dx = \left[\frac{7x^7}{7}\right]_0^2$$

Substitute the limits of the integral $x = 2$ and $x = 0$ and subtract: $F(b) - F(a)$.

$$\left[x^7\right]_0^2 = 2^7 - 0 = \mathbf{128}$$

c Integrate $4x^3$.

$$\int_1^2 4x^3 dx = \left[\frac{4x^4}{4}\right]_1^2$$

Substitute the limits of the integral $x = 2$ and $x = 1$ and subtract: $F(b) - F(a)$.

$$\left[x^4\right]_1^2 = 2^4 - 1^4 = \mathbf{15}$$

d Integrate $(2x - 1)^2$.

$$\int_2^3 (2x-1)^2\,dx = \left[\frac{(2x-1)^3}{2\times 3}\right]_2^3$$

Substitute the limits of the integral $x = 3$ and $x = 2$ and subtract: $F(b) - F(a)$.

$$\frac{1}{6}\left[(6-1)^3 - (4-1)^3\right] = \frac{1}{6}[125 - 27] = \mathbf{\frac{49}{3}}$$

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e Integrate $(x + 2)^{-2}$.

$$\int_0^4 (x+2)^{-2}\,dx = \left[\frac{(x+2)^{-1}}{-1}\right]_0^4$$

Substitute the limits of the integral $x = 4$ and $x = 0$ and subtract: $F(b) - F(a)$.

$$-\left[\frac{1}{x+2}\right]_0^4 = -\left(\frac{1}{4+2} - \frac{1}{0+2}\right) = -\left(\frac{1}{6} - \frac{1}{2}\right) = -\left(\frac{-2}{6}\right) = \mathbf{\frac{1}{3}}$$

f Integrate $\boldsymbol{x^3 - 3x^2 + 1}$.

$$\int_0^1 \left(x^3 - 3x^2 + 1\right)dx = \left[\frac{x^4}{4} - x^3 + x\right]_0^1$$

Substitute the limits of the integral $x = 1$ and $x = 0$ and subtract: $F(b) - F(a)$.

$$\left(\frac{1}{4} - 1 + 1\right) - 0 = \mathbf{\frac{1}{4}}$$

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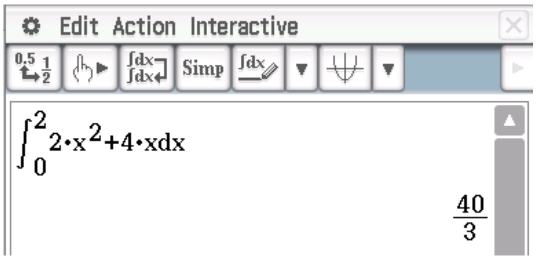
Question 4

a **ClassPad**



1 Highlight the given expression and tap **Interactive > Calculation >** $\int$.

2 In the dialogue box, tap **Definite.**

3 Enter the lower and upper limits.

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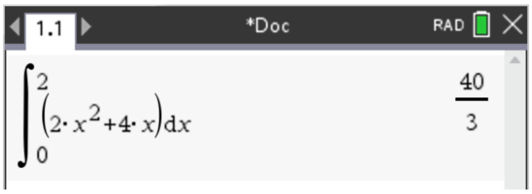

1 Press **menu > Calculus > Integral.**

2 Enter the lower and upper limits.

3 Enter the expression followed by **dx.**

$$\int_0^2 \left(2x^2+4x\right)dx=\mathbf{\frac{40}{3}}$$

b $\int_2^0 \left(2x^2+4x\right)dx=-\int_0^2 \left(2x^2+4x\right)dx=\mathbf{-\frac{40}{3}}$, using the result from part **a**.

Question 5

a Apply fundamental theorem $\frac{d}{dx}\left(\int_a^x f(t)\,dt\right)=f(x)$.

$$\boldsymbol{\int_0^x 2t^2+t-4\,dt=2x^2+x-4}$$

b Apply fundamental theorem $\frac{d}{dx}\left(\int_a^x f(t)\,dt\right)=f(x)$.

$$F'(x)=\frac{d}{dx}\int_{-2}^{x}\frac{3dt}{t^2-1}$$
$$=\boldsymbol{\frac{3}{x^2-1}}$$

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Question 6

$$\int_1^3 -5f(x)\,dx$$
$$= -5\int_1^3 f(x)\,dx$$
$$= -5 \times 3$$
$$= \mathbf{-15}$$

Question 7

$$\int_0^2 4 - 3g(x)\,dx$$
$$= \int_0^2 4\,dx - 3\int_0^2 g(x)\,dx$$
$$= [4x]_0^2 - 3 \times 5$$
$$= 4 \times 2 - 15$$
$$= \mathbf{-7}$$

Question 8 (10 marks)

(✓ = 1 mark)

a $$\int_0^2 \frac{x^2}{2}\,dx = \left[\frac{x^3}{2 \times 3}\right]_0^2 = \left[\frac{x^3}{6}\right]_0^2 \checkmark$$

$$= \frac{1}{6}\left(2^3 - 0\right)$$
$$= \frac{8}{6}$$
$$= \mathbf{\frac{4}{3}} \checkmark$$

b $\int_{-1}^{1}(3x^2+4x)\,dx=\left[x^3-2x^2\right]_{-1}^{1}$ ✓

$$=\left(1^3-2(1)^2\right)-\left((-1)^3-2(-1)^2\right)$$
$$=-1+3$$
$$=\mathbf{2}$$ ✓

c $\int_{-1}^{2}(x^2+1)\,dx=\left[\frac{x^3}{3}+x\right]_{-1}^{2}$ ✓

$$=\left(\frac{(2)^3}{3}+2\right)-\left(\frac{(-1)^3}{3}+(-1)\right)$$
$$=\frac{14}{3}-\left(\frac{-4}{3}\right)$$
$$=\frac{18}{3}$$
$$=\mathbf{6}$$ ✓

d $\int_{-2}^{3}(4x^3-3)\,dx=\left[x^4-3x\right]_{-2}^{3}$ ✓

$$=\left((3)^4-3(3)\right)-\left((-2)^4-3(-2)\right)$$
$$=72-22$$
$$=\mathbf{50}$$ ✓

e $\int_{-1}^{0}(x^2+3x+5)\,dx=\left[\frac{x^3}{3}+\frac{3x^2}{2}+5x\right]_{-1}^{0}$ ✓

$$=0-\left(\frac{-1}{3}+\frac{3}{2}-5\right)$$
$$=\mathbf{\frac{23}{6}}$$ ✓

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Question 9 (4 marks)

(✓ = 1 mark)

a Apply fundamental theorem $\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$.✓

$$\frac{d}{dx}\left(\int_0^x \sqrt{t-\pi}\,dt\right) = \sqrt{x-\pi}\ ✓$$

b Apply fundamental theorem $\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$.

$$\frac{d}{dx}\left(\int_x^0 -2t^2 + t\,dt\right) = -\frac{d}{dx}\left(\int_0^x -2t^2 + t\,dt\right)\ ✓$$

$$= -\left(-2x^2 + x\right)$$

$$= 2x^2 - x\ ✓$$

Question 10 (2 marks)

(✓ = 1 mark)

$$F'(x) = 3x + 2\frac{d}{dx}\left(\int_0^x 1 - 2t^2\,dt\right)\ ✓$$

$$= 3x + 2\left(1 - 2x^2\right)\ ✓$$

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Question 11 (3 marks)

(✓ = 1 mark)

$$\int_1^4 \left(\sqrt{x}+1\right)dx = \int_1^4 \left(x^{\frac{1}{2}}+1\right)dx \checkmark$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}}+x\right]_1^4 \checkmark$$

$$= \left(\frac{2}{3}\times 4^{\frac{3}{2}}+4\right)-\left(\frac{2}{3}\times 1^{\frac{3}{2}}+1\right)$$

$$= \frac{16}{3}+4-\frac{5}{3}$$

$$= \frac{23}{3} \checkmark$$

Question 12 (4 marks)

(✓ = 1 mark)

Use Property 3, $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$.

a $\int_0^1 x^2 dx + \int_1^5 x^2 dx = \int_0^5 x^2 dx$ ✓

b $\int_1^4 (x+1)\,dx + \int_4^7 (x+1)\,dx = \int_1^7 (x+1)\,dx$ ✓

c $\int_{-2}^0 \left(x^3-x-1\right)dx + \int_0^2 \left(x^3-x-1\right)dx = \int_{-2}^2 \left(x^3-x-1\right)dx$ ✓

d $\int_0^2 (2x+1)\,dx + \int_2^3 (2x+1)\,dx = \int_0^3 (2x+1)\,dx$ ✓

Question 13 (2 marks)

(✓ = 1 mark)

Integrate $x^2 - x^3$.

$$\int_0^1 \left(x^2 - x^3\right) dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 \checkmark$$

Substitute the limits of the integral $x = 1$ and $x = 0$ and subtract: $f(b) - f(a)$.

$$\left(\frac{1}{3} - \frac{1}{4}\right) - 0 = \mathbf{\frac{1}{12}} \checkmark$$

Question 14 (2 marks)

(✓ = 1 mark)

$$\int_1^2 \left(3x^2 + \frac{4}{x^2}\right) dx = \int_1^2 \left(3x^2 + 4x^{-2}\right) dx$$

$$= \left[x^3 + \frac{4x^{-1}}{-1}\right]_1^2 \checkmark$$

$$= \left[x^3 - \frac{4}{x}\right]_1^2$$

$$= \left(2^3 - \frac{4}{2}\right) - \left(1 - \frac{4}{1}\right)$$

$$= 6 + 3$$

$$= \mathbf{9} \checkmark$$

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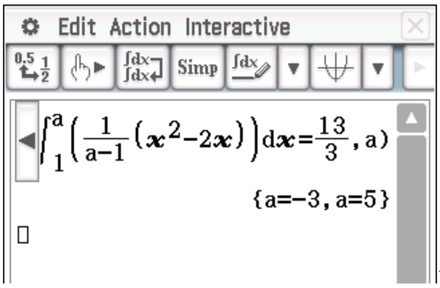
Question 15 (3 marks)

(✓ = 1 mark)

Use CAS to solve $\int_1^a \left(\frac{1}{a-1}\left(x^2-2x\right)\right)dx = \frac{13}{3}$ ✓ for a.

Casio ClassPad



✓

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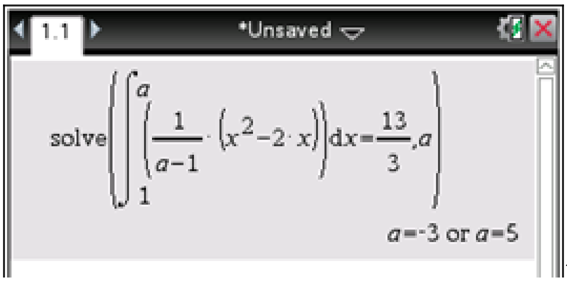

✓

Since the interval [1, a] implies that $a > 1$, $a \neq -3$, so $\boldsymbol{a = 5}$.✓

or

This can also be done by hand.

$$\frac{1}{a-1}\left[\frac{x^3}{3}-x^2\right]_1^a = \frac{13}{3} \checkmark$$

$$\frac{1}{a-1}\left(\left(\frac{a^3}{3}-a^2\right)-\left(\frac{1}{3}-1\right)\right)=\frac{13}{3}$$

$$\frac{1}{a-1}\left(\frac{a^3-3a^2+2}{3}\right)=\frac{13}{3}$$

$$a^3-3a^2+2=13a-13$$

$$a^3-3a^2-13a+15=0$$

On factorising we get,

$\boldsymbol{(a+3)(a-1)(a-5)=0}$✓

On solving,

$a = 1$ and $a = -3$ cannot be the answer. Therefore $\boldsymbol{a = 5}$.✓

Question 16 (2 marks)

(✓ = 1 mark)

$$\int_1^3 (2f(x)-3)\,dx = 2\int_1^3 f(x)\,dx - \int_1^3 3\,dx \checkmark$$

$$= 2\times 5 - [3x]_1^3$$
$$= 10-(9-3)$$

$$= \mathbf{4}\checkmark$$

Question 17 (2 marks)

(✓ = 1 mark)

Use Property 3, $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$.

$$\int_1^5 g(x)\,dx = \int_1^{12} g(x)\,dx - \int_5^{12} g(x)\,dx$$

$$= \int_1^{12} g(x)\,dx + \int_{12}^{5} g(x)\,dx \checkmark$$

$$= 5 + (-6)$$

$$= \mathbf{-1}\checkmark$$

Question 18 (2 marks)

(✓ = 1 mark)

$$\int_4^8 f(x)\,dx = F(8) - F(4)$$ ✓

$$= F(8) - (-6)$$

$$\int_4^8 f(x)\,dx = F(8) + 6$$

$$F(8) = -6 + \int_4^8 f(x)\,dx$$ ✓

Question 19 (2 marks)

(✓ = 1 mark)

Total area of rectangles is

$$1\times1^2 + 1\times2^2 + 1\times3^2 + 1\times4^2 + 1\times5^2 + 1\times6^2$$ ✓

$$= 91$$

Exact answer is $\int_0^6 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^6 = \frac{1}{3}\times6^3 = 72$ ✓

Hence $p = 91 - 72 = 19$

Percentage error is $\frac{19}{72}\times100 \approx 26$

So *p* is closer to 25.✓

EXERCISE 3.4 Area under a curve

Question 1

$$\int_1^2 \left(x^3 - 2x\right) dx = \left[\frac{1}{4}x^4 - x^2\right]_1^2$$

$$= \frac{1}{4}(2)^4 - (2)^2 - \left(\frac{1}{4}(1)^4 - (1)^2\right).$$

$$= 4 - 4 - \left(-\frac{3}{4}\right)$$

$$= \mathbf{\frac{3}{4}}$$

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Question 2

We are given $\int_0^5 g(x)\,dx = 20$.

$$\int_0^5 2g(x)\,dx = 2\int_0^5 g(x)\,dx$$
$$= 2\times 20$$
$$= 40$$

Use this answer to help determine the value of a.

$$\int_0^5 \left(2g(x)+ax\right)dx = 90$$
$$\int_0^5 2g(x)\,dx + \int_0^5 ax\,dx = 90$$
$$40 + \int_0^5 ax\,dx = 90$$

Solve $40+\int_0^5 ax\,dx = 90$ for a.

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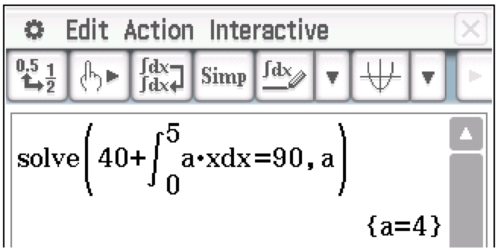

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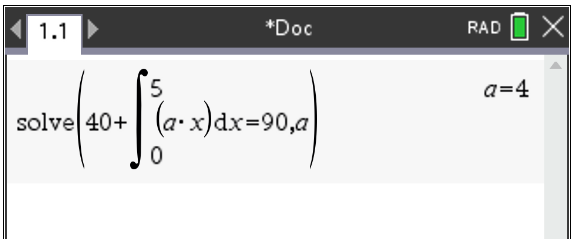


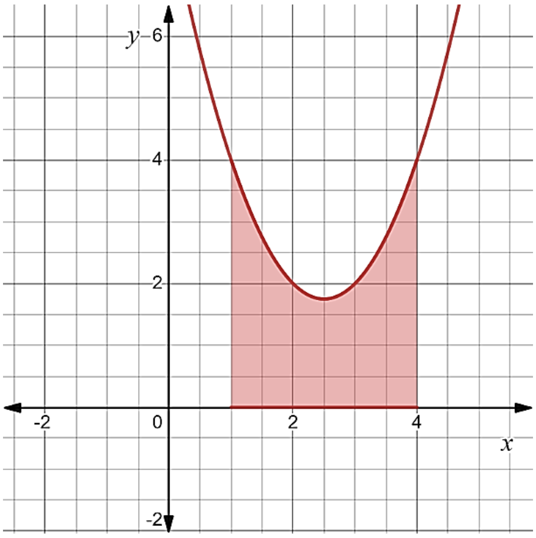
Or

$$\int_0^5 ax\,dx = 50$$
$$\left[\frac{1}{2}ax^2\right]_0^5 = 50$$
$$\frac{25}{2}a = 50$$

$\boldsymbol{a = 4}$

Question 3

a Sketch the graph showing the area required.



The integral required to find the area is $\int_1^4 \left(x^2 - 5x + 8\right) dx$.

$$\int_1^4 \left(x^2 - 5x + 8\right) dx = \left[\frac{x^3}{3} - \frac{5x^2}{2} + 8x\right]_1^4$$

$$= \left(\frac{(4)^3}{3} - \frac{5(4)^2}{2} + 8(4)\right) - \left(\frac{(1)^3}{3} - \frac{5(1)^2}{2} + 8(1)\right)$$

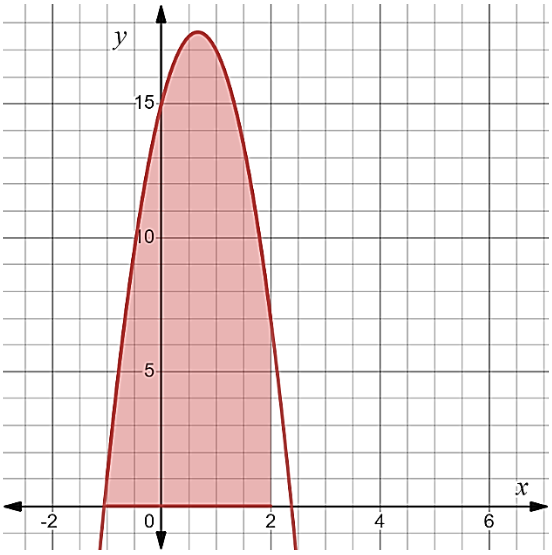
$$= \frac{128 - 240 + 192}{6} - \left(\frac{2 - 15 + 48}{6}\right)$$

$$= \frac{45}{6}$$

$$= 7.5$$

The required area is **7.5 units²**.

b Sketch the graph showing the area required.



The integral required to find the area is $\int_{-1}^{2}\left(15+8x-6x^2\right)dx$.

$$\int_{-1}^{2}\left(15+8x-6x^2\right)dx=\left[15x+\frac{8x^2}{2}-\frac{6x^3}{3}\right]_{-1}^{2}$$

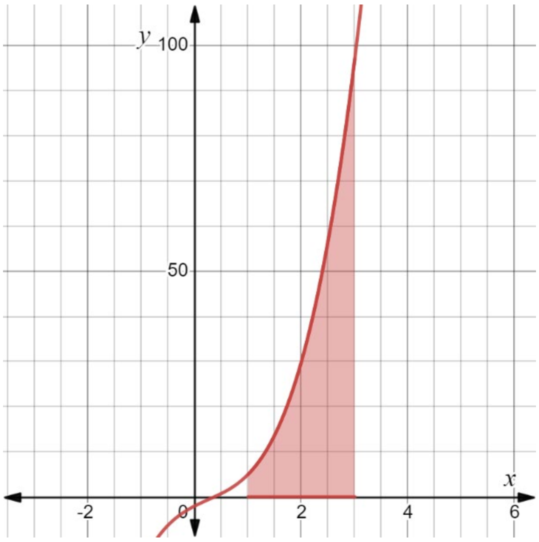
$$=\left[15x+4x^2-2x^3\right]_{-1}^{2}$$

$$=15(2)+4(2)^2-2(2)^3-\left(15(-1)+4(-1)^2-2(-1)^3\right)$$

$$=39$$

The required area is **39 units²**.

c Sketch the graph showing the area required.



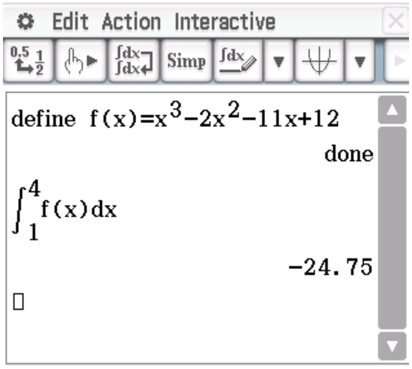
The integral required to find the area is $\int_1^3 \left(4x^3 - 3x^2 + 6x - 2\right) dx$.

$$\int_1^3 \left(4x^3 - 3x^2 + 6x - 2\right) dx = \left[x^4 - x^3 + 3x^2 - 2x\right]_1^3$$
$$= (3)^4 - (3)^3 + 3(3)^2 - 2(3) - \left((1)^4 - (1)^3 + 3(1)^2 - 2(1)\right)$$
$$= 74$$

The required area is **74 units^2**.

Question 4

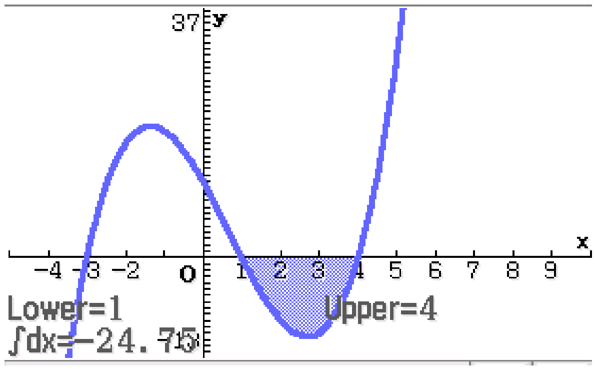
ClassPad



1 Define $f(x)$ as shown above.

2 Find the definite integral of $f(x)$ from 1 to 4.

3 The value of the positive area will be displayed.



4 To confirm this result, graph f(x).

5 Adjust the window settings to suit.

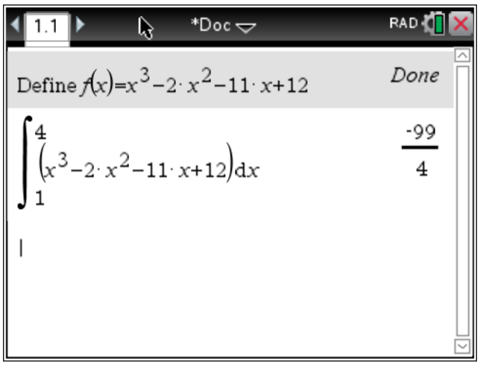
6 Tap Analysis > G-Solve > Integral > ∫dx.

7 Enter 1.

8 A dialogue box will appear with 1 in the Lower: field.

9 Enter 4 in the Upper: field.

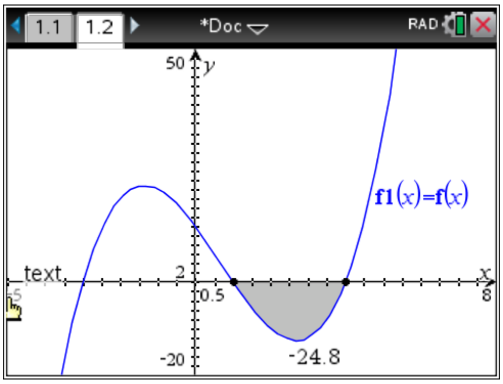
TI-Nspire



1 Define $f(x)$ as shown above.

2 Find the definite integral of $f(x)$ from 1 to 4.

3 The value of the positive area will be displayed.



5 To confirm this result, graph $f(x)$.

6 Adjust the window settings to suit.

7 Press menu > Analyze Graph > Integral.

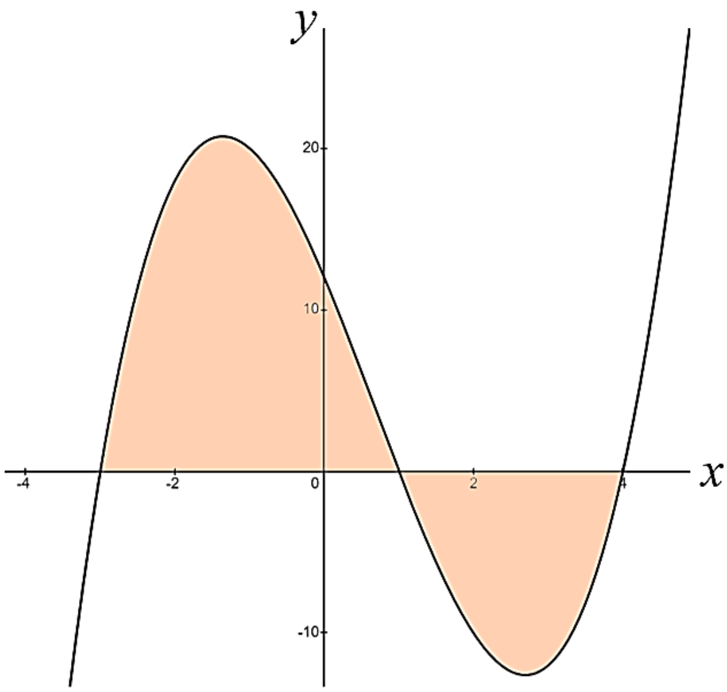
8 When prompted for the lower bound, click on 1 on the x-axis.

9 When prompted for the upper bound, click on 4 on the x-axis.

The area is 24.75 units^2.

Question 5

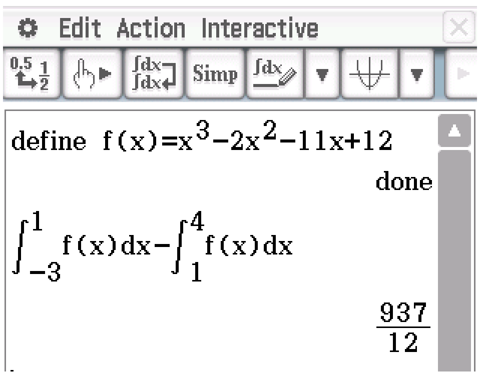
Sketch the graph to determine the required area.



The x-intercepts are $x=-3$, $x=1$, $x=4$.

The area required is $\int_{-3}^{1}\left(x^3-2x^2-11x+12\right)dx-\int_{1}^{4}\left(x^3-2x^2-11x+12\right)dx$

ClassPad



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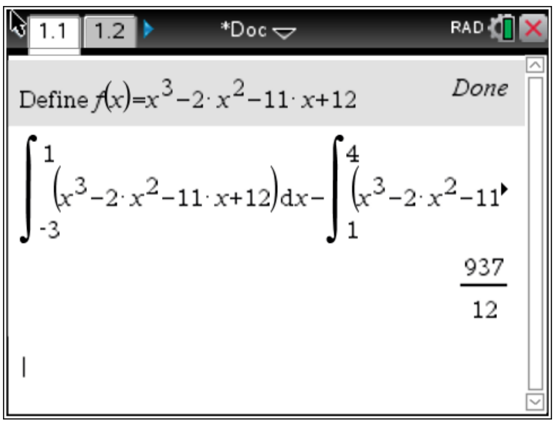

The area is $78\frac{1}{12}$ units^2.

Question 6

a $\int_0^3 f(x)\,dx$

$\int_0^3 f(x)\,dx$ is the area of the triangle from $x = 0$ to $x = 3$.

The base of the triangle is 3 and its height is 6.

The area is $\frac{1}{2} \times 3 \times 6 = 9$

$\int_0^3 f(x)\,dx = 9$

or

The gradient of the straight line from (0, 0) to (3, 6) is $\frac{6-0}{3-0} = 2$ and its equation is

$y = 2x$.

$$\begin{aligned}\int_0^3 f(x)\,dx &= \int_0^3 2x\,dx \\ &= \left[x^2\right]_0^3 \\ &= 9 - 0 \\ &= 9\end{aligned}$$

Total area is 9 units^2.

b The required answer is the area of the triangle from $x = 0$ to $x = 5$ minus the area of the triangle from $x = 5$ to $x = 7$.

The first triangle has area $\frac{1}{2} \times 5 \times 6 = 15$.

The second triangle has area $\frac{1}{2} \times 2 \times 6 = 6$.

$$\begin{aligned}\int_0^7 f(x)\,dx &= 15 - 6 \\ &= 9\end{aligned}$$

or

The gradient of the straight line from (3, 6) to (5, 0) is $\frac{0-6}{5-3} = -3$ and its equation is $y = -3x + c$.

Find c.

$0 = -3(5) + c$

c = 15

$y = -3x + c$

$$\begin{aligned}\int_3^7 f(x)\,dx &= \int_3^7 (-3x + 15)\,dx \\ &= \left[-\frac{3x^2}{2} + 15x\right]_3^7 \\ &= \left(-\frac{147}{2} + 105\right) - \left(-\frac{27}{2} + 45\right) \\ &= -60 + 60 \\ &= 0\end{aligned}$$

$$\begin{aligned}\int_0^7 f(x)\,dx &= \int_0^3 f(x)\,dx + \int_3^7 f(x)\,dx \\ &= 9 + 0 \\ &= 9\end{aligned}$$

Total area is 9 units^2.

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Question 7 (2 marks)

(✓ = 1 mark)

The area between the graph of $f(x)$, the x-axis and the lines $x = -2$ and $x = -1$ is equal to

$\int_{-2}^{-1} f(x)\,dx$.✓

Correct placement of limits of integration.✓

Question 8 (2 marks)

(✓ = 1 mark)

The x-intercept that splits the area into two regions, above and below the x-axis, is 0.✓

The area between the graph of $f(x)$, the x-axis and the lines $x = -3$ and $x = 1$ is equal to

$$\int_{-3}^{1} f(x)\,dx = \int_{-3}^{0} f(x)\,dx - \int_{0}^{1} f(x)\,dx$$

($-\int_{0}^{1} f(x)\,dx$ changing the sign makes a negative area become positive).✓

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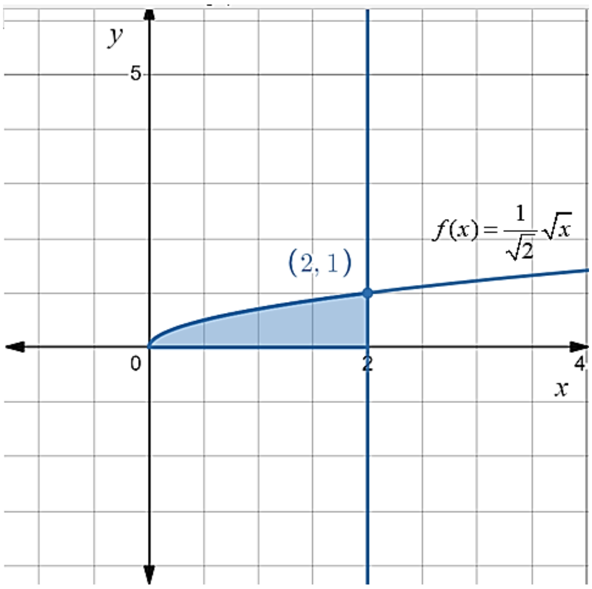
Question 9 (3 marks)

(✓ = 1 mark)

Sketch the graph showing the area required.



The integral required to find the area is $\int_0^2 f(x)\,dx = \int_0^2 \left(\frac{1}{\sqrt{2}}\sqrt{x}\right)dx$. ✓

$$\begin{aligned}
&\int_0^2 \left(\frac{1}{\sqrt{2}}\sqrt{x}\right)dx \\
&= \frac{1}{\sqrt{2}}\int_0^2 \sqrt{x}\,dx \\
&= \frac{1}{\sqrt{2}}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^2 \quad ✓ \\
&= \frac{1}{\sqrt{2}}\times\frac{2}{3}(2)^{\frac{3}{2}} \\
&= \frac{4}{3}
\end{aligned}$$

The required area is $\frac{4}{3}$ **units**^{2}. ✓

Question 10 (3 marks)

(✓ = 1 mark)

$f(x) = \sqrt{x}(1-x)$

$$A = \int_0^1 \sqrt{x}(1-x)\,dx$$
$$= \int_0^1 \left(\sqrt{x} - x\sqrt{x}\right)dx$$
$$= \int_0^1 \left(x^{\frac{1}{2}} - x^{\frac{3}{2}}\right)dx$$
$$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}\right]_0^1 \checkmark$$
$$= \frac{2}{3} - \frac{2}{5}$$
$$= \frac{10}{15} - \frac{6}{15}$$
$$= \frac{4}{15}\checkmark$$

Question 11 (4 marks)

(✓ = 1 mark)

$$\int_2^6 f(x)\,dx = \int_2^3 f(x)\,dx + \int_3^6 f(x)\,dx \checkmark$$

First find the equation of the straight line from (0, 0) to (3, 6).

Gradient of line is $\frac{6}{3} = 2$

Equation of straight line is $y = 2x$

$$\begin{aligned}\int_2^3 2x\,dx &= \left[x^2\right]_2^3 \\ &= 9 - 4 \quad \checkmark \\ &= 5\end{aligned}$$

Gradient of second line is $\frac{0-6}{5-3} = -3$

The line passes through (5, 0), so the equation is $y = -3x + 15$

$$\begin{aligned}\int_3^6 (-3x+15)\,dx &= \left[-\frac{3x^2}{2} + 15x\right]_3^6 \\ &= (-54+90) - \left(-\frac{27}{2} + 45\right) \\ &= -9 + \frac{27}{2} \quad \checkmark \\ &= \frac{9}{2} \\ &= 4.5\end{aligned}$$

Total area is 5 + 4.5 = 9.5 units^2.✓

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Question 12 (2 marks)

(✓ = 1 mark)

$\int_{-1}^{1} f(x)\,dx$ **and** $\int_{4}^{6} f(x)\,dx$ **will give negative values for area as they are below the *x*-axis.**✓ Changing the sign makes a negative area become positive.

The total area of the shaded regions is $-\int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx - \int_{4}^{6} f(x)\,dx$.✓

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Question 13 (3 marks)

(✓ = 1 mark)

$g(x) = x^2 - 4$

The x-intercept that splits the area into two regions, above and below the x-axis, is

$x^2 = 4$

$\boldsymbol{x = \pm 2}$ ✓

$x = 2$ is where the area is split.

$$A = -\int_0^2 \left(x^2 - 4\right) dx = -\left[\frac{x^3}{3} - 4x\right]_0^2 = -\left(\frac{(2)^3}{3} - 4(2)\right) = -\frac{8-24}{3} = \frac{\mathbf{16}}{\mathbf{3}}$$ ✓

This can more quickly be done using CAS.

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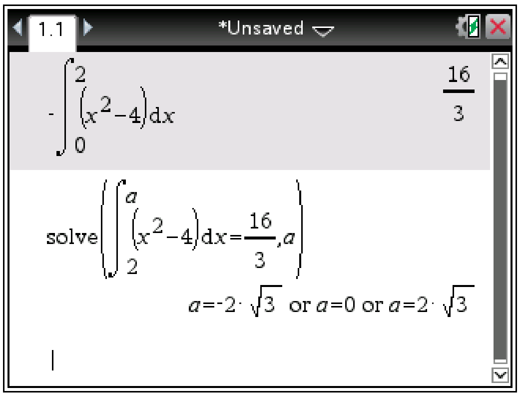
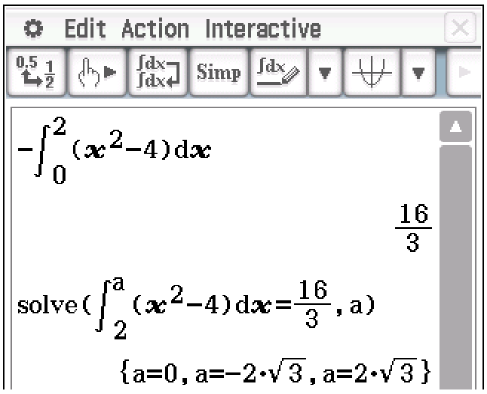


Casio ClassPad



$a = 0$ or $a = \pm 2\sqrt{3}$

As a is on positive x-axis, $\boldsymbol{a} = 2\sqrt{3}$.✓

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Question 14 (3 marks)

(✓ = 1 mark)

$$\int_{-1}^{2} 2(x+1)^3\, dx = -\left[\frac{1}{2}(x+1)^4\right]_{-1}^{2} \checkmark$$

$$= \frac{1}{2}(2+1)^4 - 0 \checkmark$$

$$= 40.5 \text{ units}^2. \checkmark$$

This can also be done using CAS.

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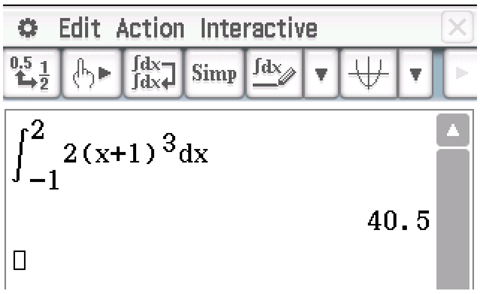

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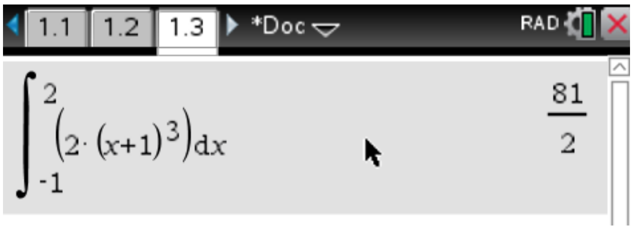
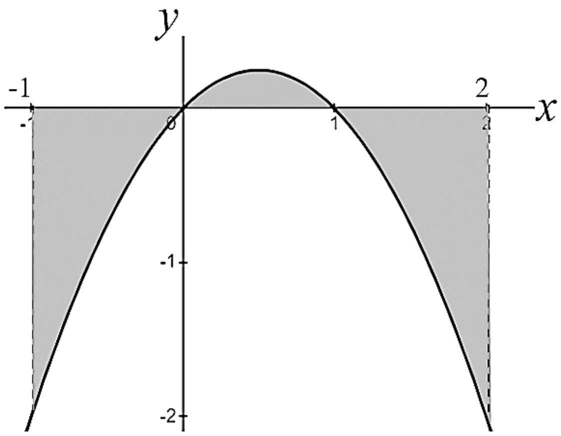


EXERCISE 3.5 Areas between curves

Question 1



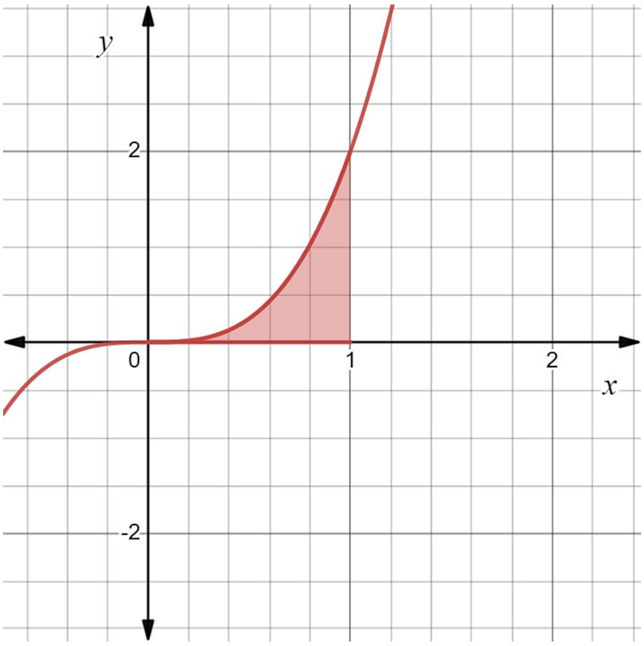
Area is $-\int_{-1}^{0} f(x)\,dx+\int_{0}^{1} f(x)\,dx-\int_{1}^{2} f(x)\,dx$

$=\int_{0}^{1} f(x)\,dx-2\int_{1}^{2} f(x)\,dx$, since $\int_{-1}^{0} f(x)\,dx=\int_{1}^{2} f(x)\,dx$

The answer is D.

Question 2

Sketch the graph showing the area required.



The required area = $\int_0^1 2x^3 dx = 2\int_0^1 x^3 dx = 2\left[\frac{x^4}{4}\right]_0^1 = 2\left[\frac{1}{4}\right] = \mathbf{\frac{1}{2}} =$ **0.5 units²**.

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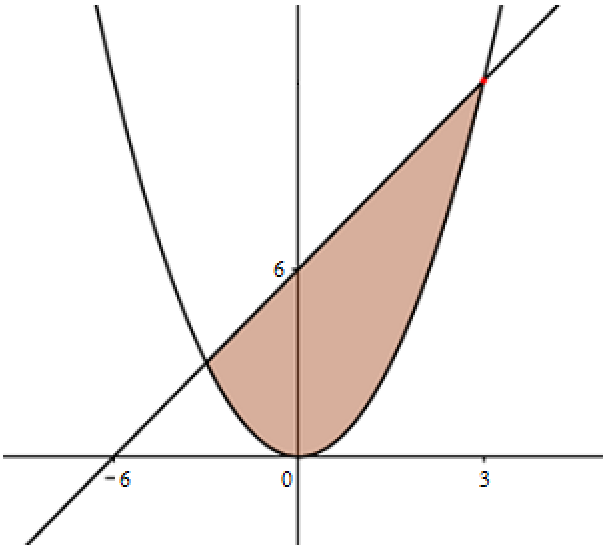
Question 3

The area enclosed by the graphs of $y = x^2$ and $y = x + 6$ is shown below.



Solving the equations simultaneously to find the intersection points,

$$x^2 = x + 6$$
$$x^2 - x - 6 = 0$$
$$(x-3)(x+2) = 0$$

$x = 3$ or $x = -2$

Between $x = -2$ and $x = 3$, the line $y = x + 6$ is the upper function. Set up the integral using $\int_a^b (\text{upper} - \text{lower})\,dx$ and simplify the terms.

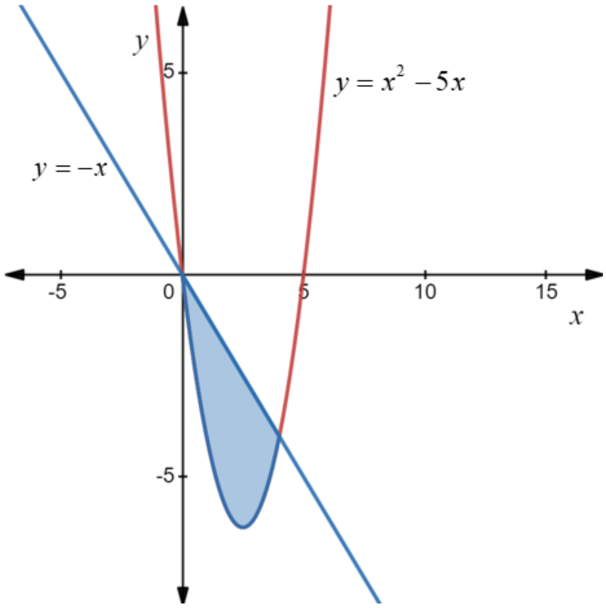
$$\int_{-2}^{3} \left((x+6) - x^2\right) dx = \int_{-2}^{3} \left(-x^2 + x + 6\right) dx = \left[\frac{-x^3}{3} + \frac{x^2}{2} + 6x\right]_{-2}^{3}$$

$$= \left(\frac{-(3)^3}{3} + \frac{(3)^2}{2} + 6(3)\right) - \left(\frac{-(-2)^3}{3} + \frac{(-2)^2}{2} + 6(-2)\right) = \frac{81}{6} + \frac{44}{6} = \frac{125}{6}$$

$\therefore$ area = $\mathbf{\frac{125}{6}}$ **units²**

Question 4

a The area enclosed by the graphs of $y = x^2 - 5x$ and $y = -x$ is shown below.



Solving the equations simultaneously to find the intersection points,

$x^2 - 5x = -x$

$x^2 - 5x + x = 0$

$x^2 - 4x = 0$

$x(x-4) = 0$

$x = 0$ or $x = 4$

Between $x = 0$ and $x = 4$, the line $y = -x$ is the upper function. Set up the integral using $\int_a^b (\text{upper} - \text{lower})\,dx$ and simplify the terms.

$$A = \int_0^4 \left(-x - \left(x^2 - 5x\right)\right)dx = \int_0^4 \left(-x^2 + 4x\right)dx$$

$$= \left[\frac{-x^3}{3} + 2x^2\right]_0^4 = \left(\frac{-(4)^3}{3} + 2(4)^2\right) - 0 = \frac{-64}{3} + 32 = \frac{32}{3}$$

$\therefore$ area = $\mathbf{\frac{32}{3}}$ **units²**

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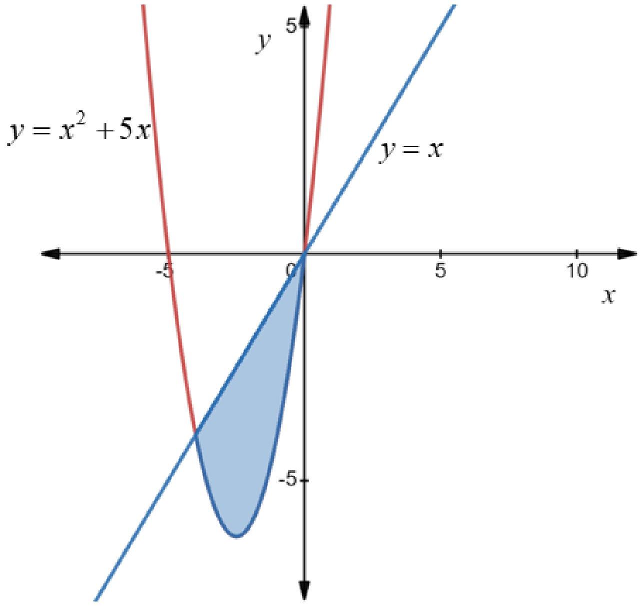
b The area enclosed by the graphs of $y = x^2 + 5x$ and $y = x$ is shown below.



Solving the equations simultaneously to find the intersection points,

$$x^2 + 5x = x$$
$$x^2 + 5x - x = 0$$
$$x^2 + 4x = 0$$
$$x(x+4) = 0$$

$x = -4$ or $x = 0$

Between $x = -4$ and $x = 0$, the line $y = x$ is the upper function. Set up the integral using $\int_a^b (\text{upper} - \text{lower})\, dx$ and simplify the terms.

$$A = A = \int_{-4}^{0} \left(x - \left(x^2 + 5x\right)\right) dx = -\int_{-4}^{0} \left(x^2 + 4x\right) dx$$

$$= -\left[\frac{x^3}{3} + 2x^2\right]_{-4}^{0} = -\left(0 - \left(\frac{(-4)^3}{3} + 2(-4)^2\right)\right) = \left(\frac{-64}{3} + 32\right) = \frac{32}{3}$$

$\therefore$ area = $\mathbf{\frac{32}{3}}$ **units²**

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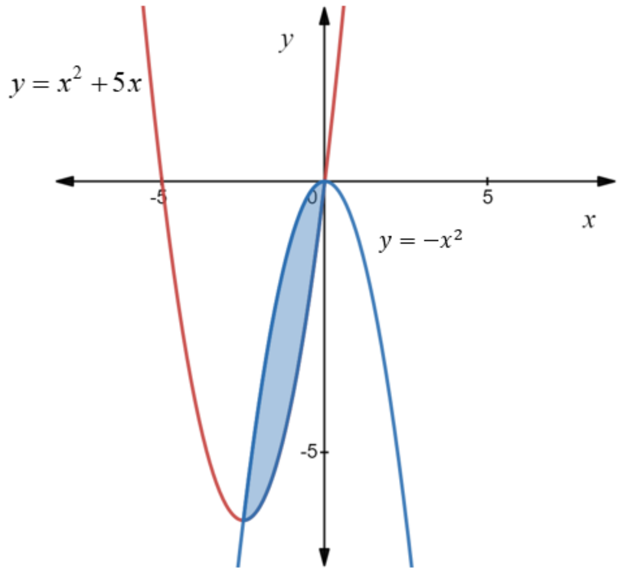
c The area enclosed by the graphs of $y = x^2 + 5x$ and $y = -x^2$ is shown below.



Solving the equations simultaneously to find the intersection points,

$$x^2 + 5x = -x^2$$
$$x^2 + 5x + x^2 = 0$$
$$2x^2 + 5x = 0$$
$$x(2x+5) = 0$$

$x = 0$ or $x = -\frac{5}{2}$

Between $x = -\frac{5}{2}$ and $x = 0$, the curve $y = -x^2$ is the upper function. Set up the integral using $\int_a^b (\text{upper} - \text{lower})\,dx$ and simplify the terms.

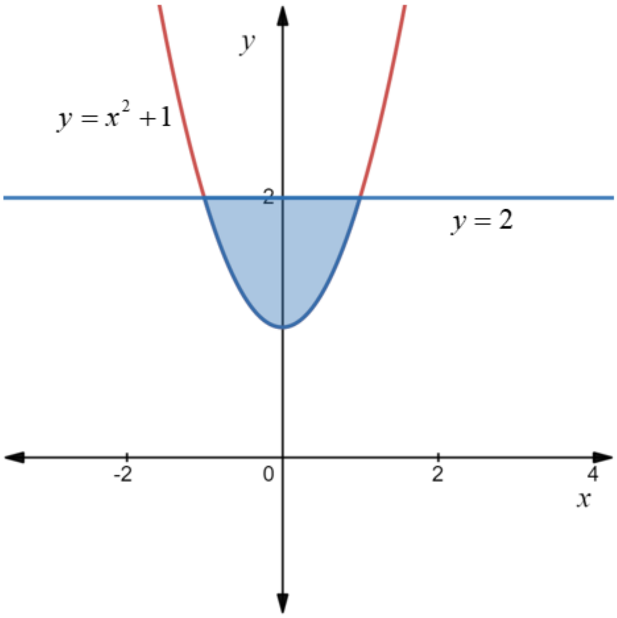
$$A = \int_{-\frac{5}{2}}^{0} \left(-x^2 - \left(x^2 + 5x\right)\right) dx = \int_{-\frac{5}{2}}^{0} \left(-2x^2 - 5x\right) dx = -\int_{-\frac{5}{2}}^{0} \left(2x^2 + 5x\right) dx$$

$$-\left[\frac{2x^3}{3} + \frac{5x^2}{2}\right]_{-\frac{5}{2}}^{0} = -\left(0 - \left(\frac{2\left(-\frac{5}{2}\right)^3}{3} + \frac{5\left(-\frac{5}{2}\right)^2}{2}\right)\right) = \frac{-125}{12} + \frac{125}{8} = \frac{125}{24}$$

$\therefore$ area = $\mathbf{\frac{125}{24}}$ **units²**

Question 5

a The area enclosed by the graphs of $y=2$ and the curve $y=x^2+1$ is shown below.



Solving the equations simultaneously to find the intersection points,

$$x^2+1=2$$
$$x^2-1=0$$
$$(x+1)(x-1)=0$$

$x=-1$ or $x=1$

Between $x=-1$ and $x=1$, the line $y=2$ is the upper function. Set up the integral using $\int_a^b (\text{upper}-\text{lower})\,dx$ and simplify the terms.

$$A=\int_{-1}^{1}\left(2-\left(x^2+1\right)\right)dx=\int_{-1}^{1}\left(-x^2+1\right)dx$$

$$=\left[\frac{-x^3}{3}+x\right]_{-1}^{1}=\left(\frac{-(1)^3}{3}+1\right)-\left(\frac{-(-1)^3}{3}+(-1)\right)=\frac{2}{3}+\frac{2}{3}=\frac{4}{3}$$

$\therefore$ area = $\mathbf{\frac{4}{3}}$ **units²**

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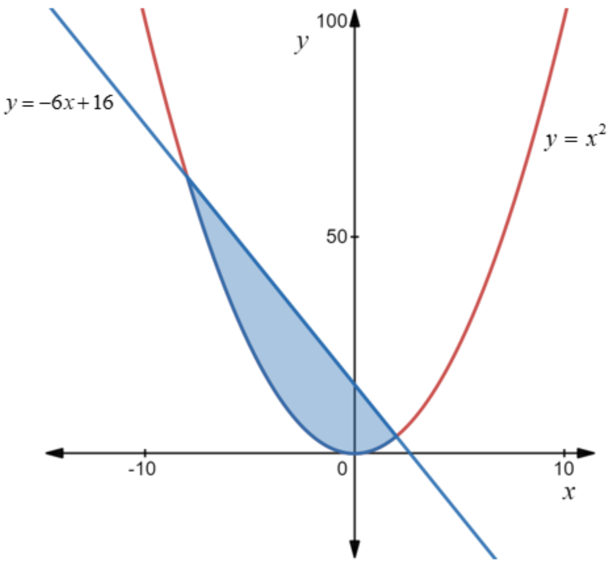
b The area enclosed by the graphs of the curve $y = x^2$ and the line $y = -6x + 16$ is shown below.



Solving the equations simultaneously to find the intersection points,

$$x^2 = -6x + 16$$
$$x^2 + 6x - 16 = 0$$
$$(x+8)(x-2) = 0$$

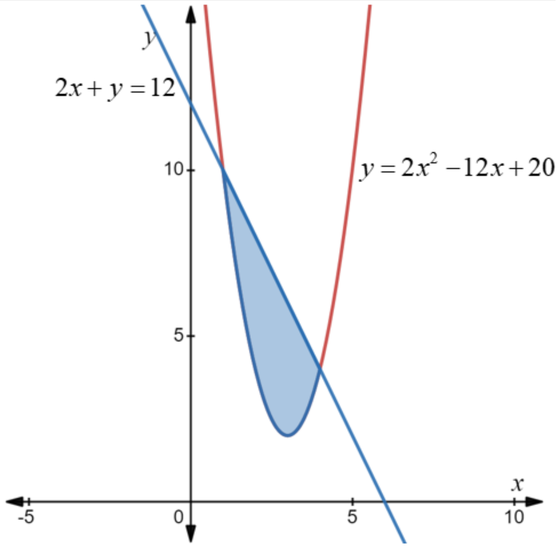
$x = -8$ or $x = 2$

Between $x = -8$ and $x = 2$, the line $y = -6x + 16$ is the upper function. Set up the integral using $\int_a^b (\text{upper} - \text{lower})\,dx$ and simplify the terms.

$$A = \int_{-8}^{2} \left((-6x+16) - x^2\right) dx = \int_{-8}^{2} \left(-x^2 - 6x + 16\right) dx = \left[\frac{-x^3}{3} - 3x^2 + 16x\right]_{-8}^{2}$$
$$= \left(\frac{-(2)^3}{3} - 3(2)^2 + 16(2)\right) - \left(\frac{-(-8)^3}{3} - 3(-8)^2 + 16(-8)\right) = \frac{52}{3} - \left(\frac{-448}{3}\right)$$
$$= \frac{500}{3}$$

$\therefore$ area = $\mathbf{\frac{500}{3}}$ **units²**

c The area enclosed by the graphs of $y = 2x^2 - 12x + 20$ and $2x + y = 12 \Rightarrow y = 12 - 2x$ is shown below.



Solving the equations simultaneously to find the intersection points,

$$2x^2 - 12x + 20 = 12 - 2x$$
$$2x^2 - 10x + 8 = 0$$
$$(x-4)(2x-2) = 0$$
$$2(x-4)(x-1) = 0$$

$x = 4$ or $x = 1$

Between $x = 1$ and $x = 4$, the line $2x + y = 12$ is the upper function. Set up the integral using $\int_a^b (\text{upper} - \text{lower})\,dx$ and simplify the terms.

$$A = \int_1^4 \left((12-2x) - \left(2x^2 - 12x + 20\right)\right) dx = \int_1^4 \left(-2x^2 + 10x - 8\right) dx = \left[\frac{-2x^3}{3} + 5x^2 - 8x\right]_1^4$$

$$= \left(\frac{-2(4)^3}{3} + 5(4)^2 - 8(4)\right) - \left(\frac{-2(1)^3}{3} + 5(1)^2 - 8(1)\right) = \frac{16}{3} - \left(-\frac{11}{3}\right)$$

$$= 9$$

$\therefore$ area = **9 units²**

Question 6 (7 marks)

(✓ = 1 mark)

a $f(x) = 3x^2 - x^3$.

$f'(x) = 6x - 3x^2$

For stationary points, $f'(x) = 0$, so solve $6x - 3x^2 = 0$ for the x-coordinates.

$6x - 3x^2 = 0$

$3x(x - 2) = 0$

$x = 0$ and 2. ✓

$f(0) = 3(0)^2 - (0)^3 = 0$

$f(2) = 3(2)^2 - (2)^3 = 4$

The stationary points are at **(0, 0)** and **(2, 4)**. ✓

b $f''(x) = 6 - 6x$

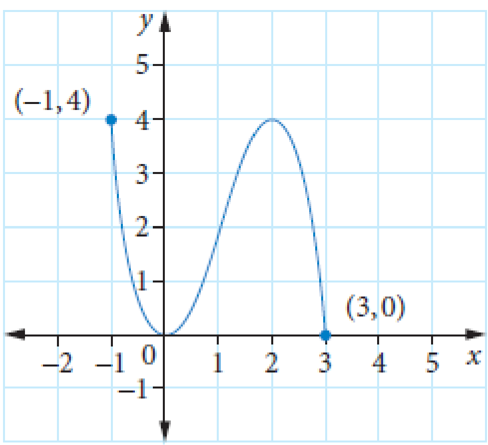
At (0, 0), $f''(0) = 6 - 6(0) = 6 > 0$ so a minimum. ✓

At (2, 4), $f''(2) = 6 - 6(2) = -6 < 0$ so a maximum. ✓

Find 2 points on the curve towards the edges of the range shown on the supplied axes.

$f(-1) = 3(-1)^2 - (-1)^3 = 4$, (–1, 4)

$f(3) = 3(3)^2 - (3)^3 = 0$, (3, 0)



✓

c The area required is from $x = -1$ to $x = 2$, but is the [3 × 4 rectangle] – [area under the curve but above the x-axis]. ✓

$$A = (3 \times 4) - \int_{-1}^{2} 3x^2 - x^3 \, dx$$

$$= 12 - \left[x^3 - \frac{x^4}{4}\right]_{-1}^{2}$$

$$= 12 - \left[(2)^3 - \frac{(2)^4}{4} - \left\{(-1)^3 - \frac{(-1)^4}{4}\right\}\right]$$

$$= 12 - \left[8 - 4 - \left(-1 - \frac{1}{4}\right)\right] \quad ✓$$

$$= 12 - \left[5 + \frac{1}{4}\right]$$

$$= 6\frac{3}{4}$$

$$\mathbf{= 6.75 \text{ units}^2}$$

Question 7 (2 marks)

(✓ = 1 mark)

From the given figure, we can see that between $x = -3$ and $x = 2$ the curve $g(x)$ is the upper function. Set up the integral using $\int_a^b (\textbf{upper} - \textbf{lower})\, dx$. ✓

$$\textbf{area} = \int_{-3}^{2} \left(g(x) - f(x)\right) dx \; ✓$$

Question 8 (2 marks)

(✓ = 1 mark)

From the given figure, we can see that between $x = -1$ and $x = 2$, the curve $f(x)$ is the upper function. Set up the integral using $\int_a^b (\textbf{upper} - \textbf{lower})\, dx$. ✓

$$\textbf{area} = \int_{-1}^{2} \left(f(x) - g(x) \right) dx = \int_{-1}^{2} f(x)\, dx - \int_{-1}^{2} g(x)\, dx \quad ✓$$

Question 9 (2 marks)

(✓ = 1 mark)

From the given figure, we can see that between $x = -1$ and $x = 1$ the curve $f(x)$ is the upper function. Set up the integral using $\int_a^b (\textbf{upper} - \textbf{lower})\, dx$. ✓

$$A_1 = \int_{-1}^{1} \left(f(x) - g(x) \right) dx$$

Between $x = 1$ and $x = 4$, the curve $g(x)$ is the upper function. Set up the integral using $\int_a^b (\text{upper} - \text{lower})\, dx$.

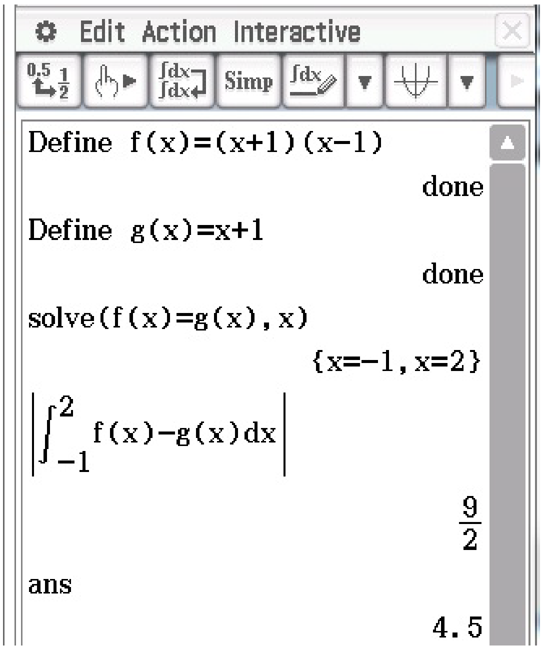
$$A_2 = \int_{1}^{4} \left(g(x) - f(x) \right) dx$$

$$\textbf{total area} = A_1 + A_2 = \int_{-1}^{1} \left(f(x) - g(x) \right) dx + \int_{1}^{4} \left(g(x) - f(x) \right) dx \; ✓$$

Question 10 (3 marks)

(✓ = 1 mark)

ClassPad



1 **Define f(x) and g(x) as shown above.✓**

2 **Solve f(x) = g(x) to determine the *x* values of the points of intersection.✓**

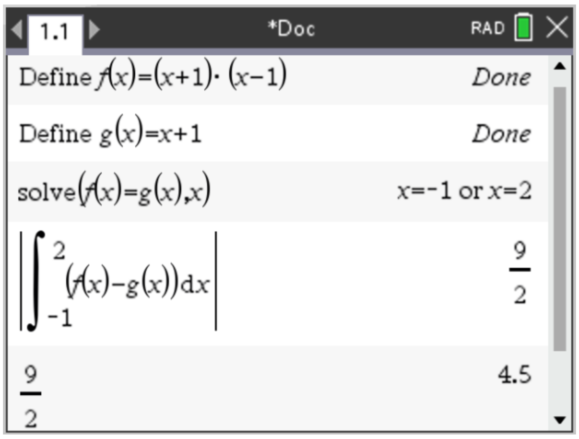
The absolute value brackets '||' around an expression make its value always positive whether it is positive or negative.

3 From the **Keyboard**, tap **Math1** or **Math2** to insert the absolute value template | |.

4 Find the definite integral of **f(x) − g(x)** from **−1 to 2**.

5 The area between the curves will be displayed.

TI-Nspire



1 **Define f(x) and g(x) as shown above.✓**

2 **Solve f(x) = g(x) to determine the *x* values of the points of intersection.✓**

3 Press the **template** key and insert the absolute value template | |.

The absolute value brackets '||' around an expression make its value always positive whether it is positive or negative.

4 Find the definite integral of **f(x) − g(x)** from **−1 to 2**.

5 The area between the curves will be displayed.

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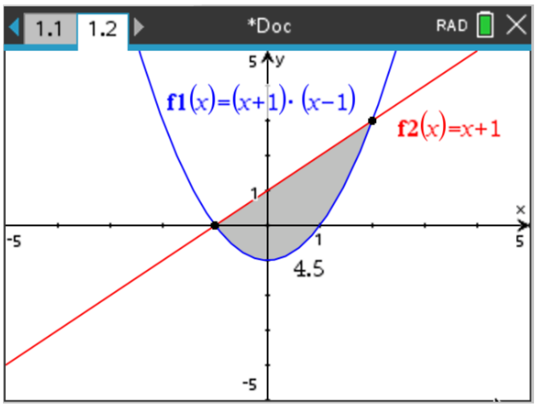


6 To confirm this result, graph **f(x)** and **g(x)**.

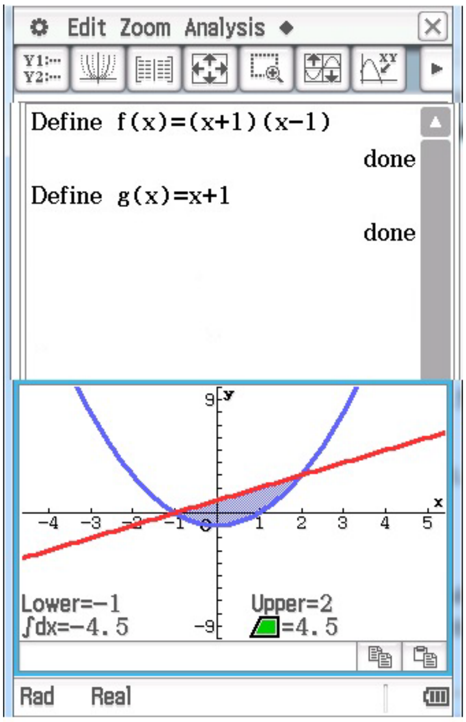
7 Adjust the window settings to suit.

8 Press **menu** > **Analyse Graph** > **Bounded Area.**

9 When prompted for the **lower bound**, click on first point of intersection on the left.

10 When prompted for the **upper bound**, click the second point of intersection.

11 The area between the first two points of intersection will be displayed.



7 To confirm this result, graph **f(x)** and **g(x)**.

8 Adjust the window settings to suit.

9 Tap **Analysis** > **G-Solve** > **Integral** > **∫dx Intersection.**

10 With the cursor on the first point of intersection, press **EXE**.

11 Press the right arrow to jump to the second point of intersection, press **EXE**.

12 The area between the first two points of intersection will be displayed.

The area is 4.5 units^2.✓

Question 11 (12 marks)

(✓ = 1 mark)

a From the given figure, we can see that between $x = -1$ and $x = 3$ the line $y = 2x+3$ is the upper function. **Set up the integral using $\int_a^b (\text{upper} - \text{lower})\, dx$ and simplify the terms.**✓

$$\int_{-1}^{3}\left((2x+3)-x^2\right)dx = \int_{-1}^{3}\left(-x^2+2x+3\right)dx = \left[\frac{-x^3}{3}+x^2+3x\right]_{-1}^{3} \checkmark$$

$$= \left(\frac{-(3)^3}{3}+(3)^2+3(3)\right)-\left(\frac{-(-1)^3}{3}+(-1)^2+3(-1)\right) = 9-\left(-\frac{5}{3}\right)$$

$$= \frac{32}{3}$$

$\therefore$ area = $\mathbf{\frac{32}{3}}$ **units²**✓

b From the given figure, we can see that between $x = -4$ and $x = 3$ the line $y = 2x+1$ is the upper function. **Set up the integral using $\int_a^b (\text{upper} - \text{lower})\, dx$ and simplify the terms.**✓

$$\int_{-4}^{3}\left((2x+1)-(x-3)\right)dx = \int_{-4}^{3}(x+4)\,dx = \left[\frac{x^2}{2}+4x\right]_{-4}^{3} \checkmark$$

$$= \left(\frac{(3)^2}{2}+4(3)\right)-\left(\frac{(-4)^2}{2}+4(-4)\right) = \frac{33}{2}-(-8)$$

$$= \frac{49}{2}$$

$\therefore$ area = $\mathbf{\frac{49}{2}}$ **units²**✓

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c From the given figure, we can see that between $x = 0$ and $x = 2$ the curve $y = 6 - x^2$ is the upper function. **Set up the integral using $\int_a^b (\text{upper} - \text{lower})\, dx$ and simplify the terms.**✓

$$\int_0^2 \left(\left(6 - x^2\right) - x\right) dx = \int_0^2 \left(-x^2 - x + 6\right) dx = \left[-\frac{x^3}{3} - \frac{x^2}{2} + 6x\right]_0^2 \checkmark$$

$$= \left(-\frac{(2)^3}{3} - \frac{(2)^2}{2} + 6(2)\right) - 0$$

$$= \frac{22}{3}$$

$\therefore$ area = $\mathbf{\frac{22}{3}}$ **units²**✓

d From the given figure, we can see that between $x = 0$ and $x = 3$ the curve $y = 18 - x^2$ is the upper function. **Set up the integral using $\int_a^b (\text{upper} - \text{lower})\, dx$ and simplify the terms.**✓

$$\int_0^3 \left(\left(18 - x^2\right) - x^2\right) dx = \int_0^3 \left(-2x^2 + 18\right) dx = \left[\frac{-2x^3}{3} + 18x\right]_0^3 \checkmark$$

$$= \left(\frac{-2(3)^3}{3} + 18(3)\right) = \frac{-54}{3} + 54$$

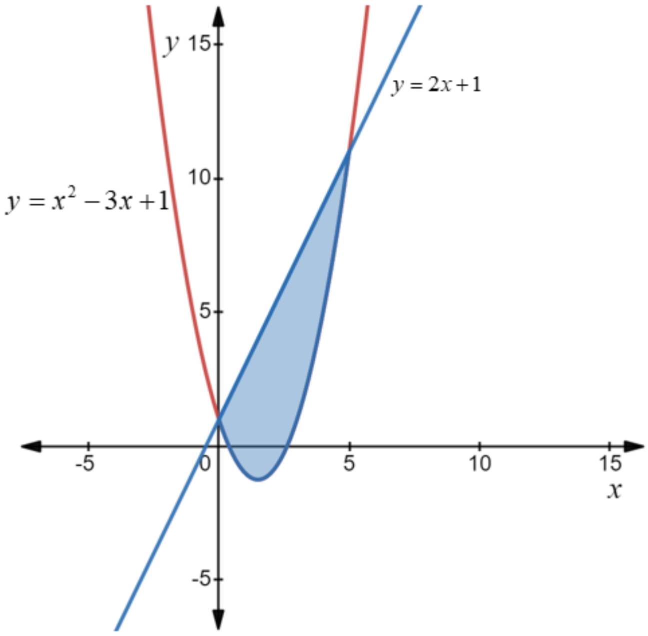
$$= 36$$

$\therefore$ area = **36 units²**✓

Question 12 (3 marks)

(✓ = 1 mark)

The area enclosed by the graphs of $y = x^2 - 3x + 1$ and $y = 2x + 1$ is shown below.



Solving the equations simultaneously to find the intersection points,

$x^2 - 3x + 1 = 2x + 1$

$x^2 - 5x = 0$

$x(x-5) = 0$

$\boldsymbol{x = 0}$ **or** $\boldsymbol{x = 5}$✓

Between $x = 0$ and $x = 5$, the line $y = 2x + 1$ is the upper function. Set up the integral using $\int_a^b (\text{upper} - \text{lower})\,dx$ and simplify the terms.

$$\int_0^5 \left((2x+1) - (x^2 - 3x + 1)\right) dx = \int_0^5 \left(-x^2 + 5x\right) dx = \left[\frac{-x^3}{3} + \frac{5x^2}{2}\right]_0^5 \checkmark$$

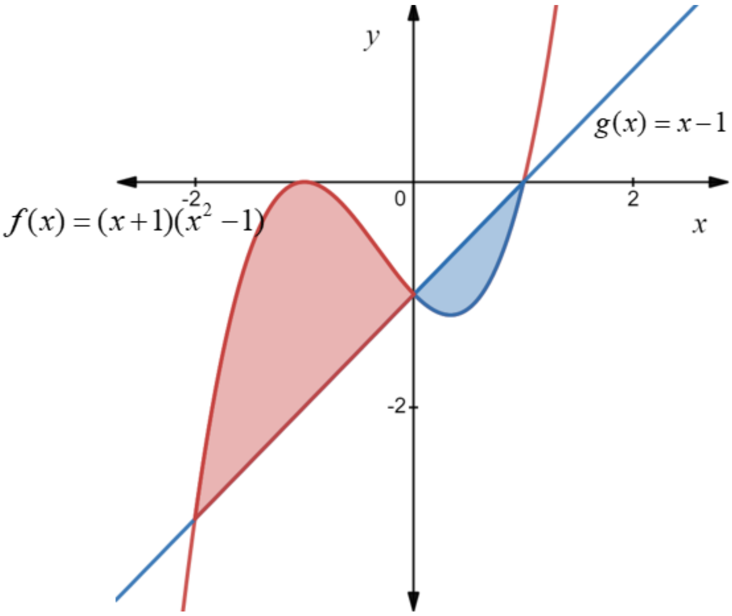
$$= \left(\frac{-(5)^3}{3} + \frac{5(5)^2}{2}\right)$$

$$= \frac{125}{6} \checkmark$$

Question 13 (3 marks)

(✓ = 1 mark)

The area enclosed by the graphs of $f(x)=(x+1)(x^2-1)$ and $g(x)=x-1$ is shown below.



Solving the equations simultaneously to find the intersection points,

$$(x+1)(x^2-1)=x-1$$
$$x^3+x^2-2x=0$$
$$x(x^2+x-2)=0$$

$$\boldsymbol{x(x-1)(x+2)=0}$$ ✓

$x = 0$ or -2 or 1

Between $x = -2$ and $x = 0$, the curve $f(x)=(x^2-1)(x+1)$ is the upper function. Set up the integral using $\int_a^b(\text{upper}-\text{lower})\,dx$.

$$\int_{-2}^{0}\left((x+1)(x^2-1)-(x-1)\right)dx=\int_{-2}^{0}\left(x^3+x^2-2x\right)dx$$

Between $x = 0$ and $x = 1$, the line $g(x)=x+1$ is the upper function. Set up the integral using $\int_a^b(\text{upper}-\text{lower})\,dx$.

$$\int_0^1 \left(\left(x-1\right)-\left(x+1\right)\left(x^2-1\right)\right)dx = \int_0^1 \left(-x^3-x^2+2x\right)dx$$

Total area $= \int_{-2}^{0}\left(x^3+x^2-2x\right)dx - \int_0^1\left(x^3+x^2-2x\right)dx$ ✓

$$= \left[\frac{x^4}{4}+\frac{x^3}{3}-x^2\right]_{-2}^{0} - \left[\frac{x^4}{4}+\frac{x^3}{3}-x^2\right]_0^1$$

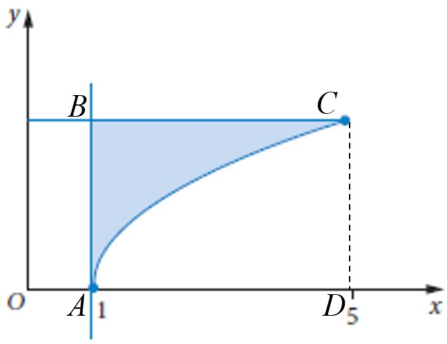
$$= \frac{32}{12}-\left(-\frac{5}{12}\right)$$

$$= \mathbf{\frac{37}{12}}$$

= **3.083** units^2 ✓

Question 14 (3 marks)

(✓ = 1 mark)



Area required is area of rectangle *ABCD* minus area between the parabola and the *x*-axis from $x = 1$ to $x = 5$.✓

$f(5) = 2$, so area of rectangle is $4 \times 2 = 8$.

$$\int_1^5 \sqrt{x-1}\,dx = \left[\frac{(x-1)^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^5$$

$$= \frac{16}{3} - 0$$

$$= \mathbf{5\frac{1}{3}}$$ ✓

$\therefore$ area = $\mathbf{8 - 5\frac{1}{3} = 2\frac{2}{3} = \frac{8}{3}}$ **units²** ✓

Question 15 (4 marks)

(✓ = 1 mark)

Solving $f(x) = x^3 - ax$ *and* $g(x) = ax$

$$x^3 - ax = ax$$
$$x^3 - 2ax = 0$$
$$x(x^2 - 2a) = 0$$

$x = 0$ or $x = \sqrt{2a}$

i.e., $\boldsymbol{m = \sqrt{2a}}$ **(since $\boldsymbol{m > 0}$)**✓

From the given figure, we can see that between $x = 0$ and $x = m$ the curve $g(x) = ax$ is the upper function. Set up the integral using $\int_a^b (\text{upper} - \text{lower})\,dx$ and simplify the terms.

$$\int_0^m \left(ax - \left(x^3 - ax\right)\right)dx = 64$$
$$\int_0^m \left(2ax - x^3\right)dx = 64$$
$$\left[ax^2 - \frac{x^4}{4}\right]_0^m = 64$$
$$\left(am^2 - \frac{m^4}{4}\right) - 0 = 64$$

$$\boldsymbol{am^2 - \frac{m^4}{4} = 64}$$ ✓

Substituting $m = \sqrt{2a}$,

$$2a^2 - \frac{4a^2}{4} = 64$$
$$a^2 = 64$$

$\boldsymbol{a = 8}$ (since $a > 0$) ✓

Since $a = 8$, $m = \sqrt{2a} = \sqrt{16} = \mathbf{4}$.✓

EXERCISE 3.6 Straight line motion

Question 1

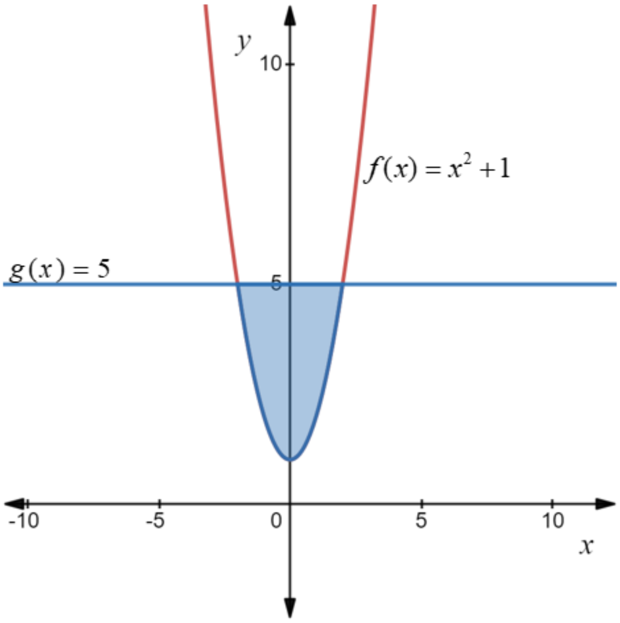
The area under the curve $y = x^2 + 1$ from $x = 0$ to $x = 3$ is

$$\int_0^3 \left(x^2 + 1\right) dx = \left[\frac{x^3}{3} + x\right]_0^3 = \left(\frac{(3)^3}{3} + 3\right) - 0 = 12$$

The correct response is **E**.

Question 2

The area enclosed by the graphs of $f(x) = x^2 + 1$ and $g(x) = 5$ is shown below.



Solving the equations simultaneously to find the intersection points,

$$x^2 + 1 = 5$$
$$x^2 - 4 = 0$$
$$(x-2)(x+2) = 0$$

$x = –2$ or $x = 2$

Between $x = –2$ and $x = 2$, the line $g(x) = 5$ is the upper function. Set up the integral using $\int_a^b (\text{upper} - \text{lower})\, dx$ and simplify the terms.

$$\int_{-2}^{2} \left(5 - \left(x^2 + 1\right)\right) dx = \int_{-2}^{2} \left(-x^2 + 4\right) dx = \left[\frac{-x^3}{3} + 4x\right]_{-2}^{2}$$

$$= \left(\frac{-(2)^3}{3} + 4(2)\right) - \left(\frac{-(-2)^3}{3} + 4(-2)\right)$$

$$= \frac{32}{3}$$

The correct response is **B**.

Question 3

$$v = \int a(t)\,dt$$

$$= 2t^2 + t + c$$

$$t = 0, v = 0 \Rightarrow c = 0$$

$$\boldsymbol{v = 2t^2 + t}$$

Question 4

a

$$v = \int a(t)\,dt$$

$$= 3t - t^2 + c$$

$$t = 3, v = 2$$

$$2 = 3(3) - (3)^2 + c$$

$$c = 2$$

$$\boldsymbol{v = 3t - t^2 + 2}$$

b

$$x = \int_1^2 \left(3t - t^2 + 2\right) dt$$

$$= \left[2t + \frac{3}{2}t^2 - \frac{1}{3}t^3\right]_1^2$$

$$= \left(2(2) + \frac{3}{2}(2)^2 - \frac{1}{3}(2)^3\right) - \left(2(1) + \frac{3}{2}(1)^2 - \frac{1}{3}(1)^3\right)$$

$$= 4\frac{1}{6}$$

Change in displacement is $\mathbf{4\frac{1}{6} = \frac{25}{6}}$ **m**.

c Since the particle's velocity is positive between $t = 1$ and $t = 2$, the distance travelled is $\mathbf{4\frac{1}{6}}$ **m**.

Question 5

$$x(t)=\int\left(-t^2+t\right)dt$$
$$=-\frac{1}{3}t^3+\frac{1}{2}t^2+c$$
$$t=2, x=0$$
$$0=-\frac{1}{3}(2)^3+\frac{1}{2}(2)^2+c$$
$$c=\frac{2}{3}$$
$$\mathbf{x(t)=-\frac{1}{3}t^3+\frac{1}{2}t^2+\frac{2}{3}=-\frac{t^3}{3}+\frac{t^2}{2}+\frac{2}{3}}$$

Question 6

$$x(t)=\int\left(3t^2+4t^3\right)dt$$
$$=t^3+t^4+c$$
$$t=0,\ x=0\Rightarrow c=0$$
$$\mathbf{x(t)=t^3+t^4}$$

Question 7 (2 marks)

(✓ = 1 mark)

$$v(t)=\int 4t\ dt$$
$$=2t^2+c$$
$$t=2, v=0$$
$$0=2(2)^2+c$$
$$\mathbf{c=-8}\ ✓$$
$$\mathbf{v(t)=2t^2-8}\ ✓$$

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Question 8 (2 marks)

(✓ = 1 mark)

$$v(t) = \int (3 - 4t)\,dt$$

$$= 3t - 2t^2 + c$$

$$t = 0, v = 0 \Rightarrow c = 0$$

$$v(t) = 3t - 2t^2$$

✓(showing anti-differentiation twice)

$$x(t) = \int (3t - 2t^2)\,dt$$

$$= \frac{3}{2}t^2 - \frac{2}{3}t^3 + c$$

$$t = 0, x = 0 \Rightarrow c = 0$$

$$\boldsymbol{x(t) = \frac{3}{2}t^2 - \frac{2}{3}t^3 = \frac{3t^2}{2} - \frac{2t^3}{3}}$$ ✓

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Question 9 (5 marks)

(✓ = 1 mark)

a $v(t) = \int 0.4\,dt$

$= 0.4t + c$

$t = 0, v = 8 \Rightarrow \boldsymbol{c = 8}$ ✓

$\boldsymbol{v(t) = 0.4t + 8}$ ✓

b $x(t) = \int (0.4t + 8)\,dt$

$= 0.2t^2 + 8t + c$

$t = 0, x = 0 \Rightarrow \boldsymbol{c = 0}$ ✓

$\boldsymbol{x(t) = 0.2t^2 + 8t}$ ✓

c $x(t) = 0.2t^2 + 8t$

$x(10) = 0.2(10)^2 + 8(10) = 100$ ✓

The displacement after 10 seconds is 100 m.

Question 10 [SCSA MM2021 Q14] (5 marks)

(✓ = 1 mark)

$$x(t)=\frac{5t\left(t^2-15t+48\right)}{6}, t\geq 0$$

$$v(t)=\frac{dx}{dt}=\frac{5t^2-50t+80}{2}$$

$$a(t)=\frac{dv}{dt}=5t-25$$

$$5t-25=0$$

$$\therefore t=5$$

$$\text{Distance travelled}=\int_0^5\left|\frac{5t^2-50t+80}{2}\right|dt$$

$$=\frac{245}{3}$$

$$\approx 81.7 \text{ metres}$$

determines an expression for velocity✓

determines an expression for acceleration✓

equates acceleration to zero and determines *t*✓

shows integration expression for distance travelled✓

determines how far the power boat has travelled✓

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Question 11 [SCSA MM2021 Q17] (8 marks)

(✓ = 1 mark)

a $a(t) = kt^2 - 23t + 20k$

$v(t) = \dfrac{kt^3}{3} - \dfrac{23t^2}{2} + 20kt + c$

$c = 0$ since $v(0) = 0$

$18 = \dfrac{8k}{3} - 46 + 40k$

$\therefore k = 1.5$

determines an expression for the velocity including determining *c* = 0✓

substitutes *t* = 2, *v* = 18✓

correctly determines *k*✓

b $v(t) = 0.5t^3 - 11.5t^2 + 30t$

Cable car stops at the resort station ⇒ velocity = 0

$0 = 0.5t^3 - 11.5t^2 + 30t$

$\therefore t = 0, 3, 20$

It takes 20 – 3 = 17 minutes to reach the resort station.

equates the velocity to zero✓

solves for *t* ✓

states the time taken✓

c distance travelled = $\int_3^{10} \left|0.5t^3 - 11.5t^2 + 30t\right| dt$

= 1124.958

≈1125 metres

The cable car is 1125 metres away the mountain station.

writes an expression that can be used to determine position✓

determines the position of the cable car✓

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Question 12 (7 marks)

(✓ = 1 mark)

a velocity $\boldsymbol{v(t) = \int \left(\frac{t}{2} - 4\right) dt}$ ✓

$$= \frac{t^2}{4} - 4t + c$$

$t = 0, v = 16$, hence $\boldsymbol{c = 16}$ ✓

$$v(t) = \frac{t^2}{4} - 4t + 16$$

$$v(4) = \frac{4^2}{4} - 4(4) + 16 = 4$$

The velocity after 4 seconds is 4 m/s✓

b **Find when the velocity is zero.**✓

$$v(t) = 0 \Rightarrow \frac{t^2}{4} - 4t + 16 = 0$$

$$t^2 - 16t + 64 = 0$$

$$(t-8)^2 = 0$$

$\boldsymbol{t = 8}$ ✓

$$\boldsymbol{x(t) = \int_0^8 \left(\frac{t^2}{4} - 4t + 16\right) dt}$$ ✓

$$= \left[\frac{t^3}{12} - 2t^2 + 16t\right]_0^8$$

$$= \frac{128}{3}$$

The distance travelled is $\boldsymbol{\frac{128}{3}}$ **m**✓

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Cumulative examination: Calculator-free

Question 1 (2 marks)

(✓ = 1 mark)

$$f'(x) = \sqrt{x^2 + 4}$$ ✓

$$f'(2) = \sqrt{2^2 + 4} = \sqrt{8}$$

$$= 2\sqrt{2}$$ ✓

Question 2

(✓ = 1 mark)

$$\int_1^2 \left(3x^2 - \frac{1}{x^2}\right) dx = \int_1^2 \left(3x^2 - x^{-2}\right) dx$$

$$= \left[x^3 + \frac{1}{x}\right]_1^2$$ ✓

$$= \left(2^3 + \frac{1}{2}\right) - \left(1^3 + \frac{1}{1}\right)$$

$$= 8 + \frac{1}{2} - 2$$

$$== \frac{13}{2}$$ ✓

Question 3 (2 marks)

(✓ = 1 mark)

$$f(x) = \int \left(x^2 - 10x - x^{-\frac{1}{2}} + 1 \right) dx$$

$$= \frac{1}{3}x^3 - 5x^2 - 2x^{\frac{1}{2}} + x + c$$

$$f(4) = \frac{64}{3}$$

$$\frac{64}{3} = \frac{1}{3}(4)^3 - 5(4)^2 - 2(4)^{\frac{1}{2}} + 4 + c$$

$$\frac{64}{3} = \frac{64}{3} - 80 - 4 + 4 + c$$

hence $\boldsymbol{c = 80}$ ✓

$$\boldsymbol{f(x) = \frac{1}{3}x^3 - 5x^2 - 2x^{\frac{1}{2}} + x + 80} \checkmark$$

or

$$\boldsymbol{f(x) = \frac{1}{3}x^3 - 5x^2 - 2\sqrt{x} + x + 80} \checkmark$$

Question 4 [SCSA MM2021 Q3] (7 marks)

(✓ = 1 mark)

Firstly, note that

$$f'(x) = 3ax^2 + 2bx + c$$

and

$$f''(x) = 6ax + 2b$$

Given that there is an inflection point at $x = 0$ it follows that $f''(0) = 0$. Hence

$$0 = 2b$$

$$b = 0$$

Given that there is a turning point at $x = 1$ it follows that $f'(1) = 0$. Hence

$$0 = 3a + c$$

$$c = -3a$$

Given that there is a x-intercept at $x = 1$ it follows that $f(1) = 0$. Hence

$$0 = a - 3a + d$$

$$d = 2a$$

Finally, given that the area of the shaded region is $\frac{3}{2}$ it follows that $\int_0^1 f(x)\,dx = \frac{3}{2}$.

$$a\int_0^1 \left(x^3 - 3x + 2\right) dx = \frac{3}{2}$$

$$a\left[\frac{x^4}{4} - \frac{3x^2}{2} + 2x\right]_0^1 = \frac{3}{2}$$

$$\frac{3a}{4} = \frac{3}{2}$$

$$a = 2$$

Hence $a = 2$, $b = 0$, $c = -6$ and $d = 4$.

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states the first and second derivatives of *f*✓

recognises that $f''(0) = 0$ and hence that $b = 0$✓

recognises that $f'(1) = 0$ and hence that $c = -3a$✓

recognises that $f(1) = 0$ and hence that $d = 2a$ (or $d = -a - c$)✓

recognises that $\int_0^1 f(x)\,dx = \frac{3}{2}$✓

evaluates definite integral to determine that $a = 2$✓

solves for the values of *c* and *d*✓

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Question 5 [SCSA MM2016 Q7] (7 marks)

(✓ = 1 mark)

a $36+\frac{\pi 4^2}{4}+4\times 2+\frac{1}{2}2^2=46+4\pi$

determines areas of two rectangles✓

determines area of triangle and sector✓

adds areas together✓

b $46+4\pi-\left(\frac{1}{2}2^2+\left(4\times 6-\frac{\pi 4^2}{4}\right)\right)=20+8\pi$

determines area under axis✓

uses signed areas to find net result✓

c $6(\alpha-18)+26-4\pi=46+4\pi$

$6(\alpha-18)=20+8\pi$

$\boldsymbol{\alpha=\frac{20+8\pi}{6}+18}$

$\alpha=\frac{128+8\pi}{6}$

$\alpha=\frac{64+4\pi}{3}$

determines a value so that signed areas balance✓

derives an expression for α ✓

Question 6 [SCSA MM2021 Q5] (6 marks)

(✓ = 1 mark)

a $x^2 - x + 3 = x + 3$

$x^2 - 2x = 0$

$x(x-2) = 0$

$\therefore x = 0, 2$

$\int_0^2 \left(\left(x+3\right)-\left(x^2 - x + 3\right)\right) dx$

$= \int_0^2 \left(2x - x^2\right) dx$

$= \left[x^2 - \frac{x^3}{3}\right]_0^2$

$= 4 - \frac{8}{3}$

$= \frac{4}{3} \text{ units}^2$

determines *x* coordinates of the points of intersection✓

states correct integral for area✓

evaluates integral✓

determines correct area✓

b Both graphs from part **a** have been vertically translated down 5 units.

The shape of both graphs is unchanged.

Therefore, the area between them remains unchanged.

states both graphs have been translated in the same direction by the same amount✓

states both graphs retain the same shape✓

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Cumulative examination: Calculator-assumed

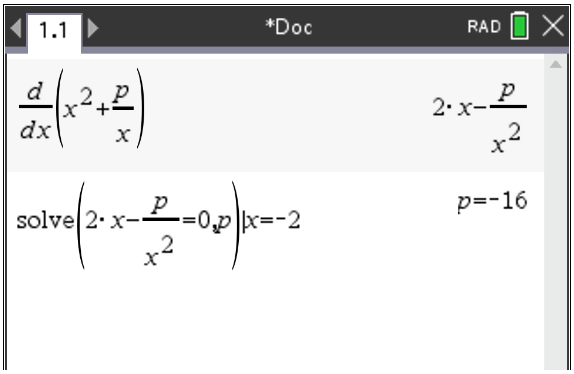
Question 1 (2 marks)

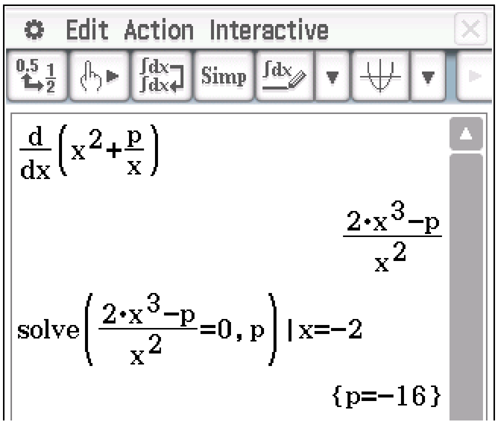
(✓ = 1 mark)

$$f(x) = x^2 + \frac{p}{x} = x^2 + px^{-1}$$

$$f'(x) = 2x - px^{-2} = 2x - \frac{p}{x^2}$$

Solve $2x - \frac{p}{x^2} = 0$ when $x = -2$ to determine the value of p.



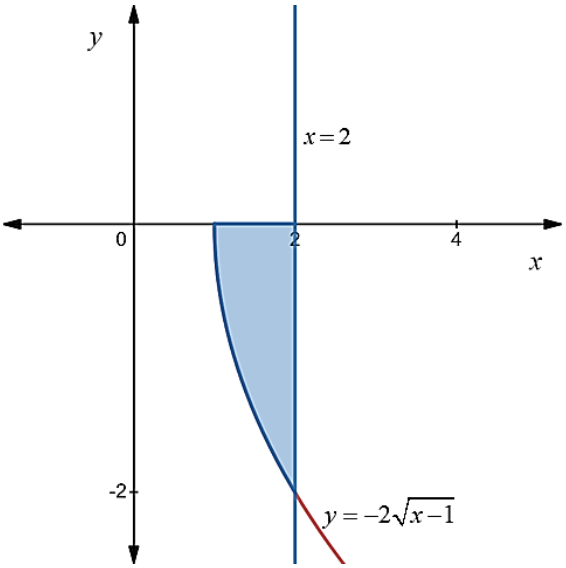


$\boldsymbol{p = -16}$

Question 2 (3 marks)

(✓ = 1 mark)

The shaded region is the area enclosed between the graph of $y=-2\sqrt{x-1}$, the x-axis and the line $x = 2$ and is shown below.



$y=0 \Rightarrow -2\sqrt{x-1}=0 \Rightarrow \boldsymbol{x=1}$ ✓

Required area = $\boldsymbol{-\int_1^2 -2\sqrt{x-1}\,dx = 2\int_1^2 \sqrt{x-1}\,dx}$ ✓

$= \boldsymbol{\frac{4}{3}}$ **units²** ✓

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Question 3 [SCSA MM2017 Q20] (9 marks)

(✓ = 1 mark)

a $a(0) = -13$ cm/s^2

determines initial acceleration✓

b $a(t) = pt - 13$

$v(t) = \frac{pt^2}{2} - 13t + c$

Since $v(0) = 5$, $c = 5$

$x(t) = \frac{pt^3}{6} - \frac{13t^2}{2} + 5t + k$

When $t = 2$: $-50 = \frac{8p}{6} - 16 + k$

When $t = 6$: $178 = 36p - 204 + k$

Solving gives: $p = 12$ and $k = -50$

determines $v(t)$ and determines the constant c✓

determines $x(t)$✓

finds the two displacement equations✓

correctly determines p✓

c $0 = 6t^2 - 13t + 5$

$t = \frac{1}{2}, \frac{5}{3}$ seconds

equates velocity to zero✓

solves to give both values of t✓

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d $47 = 12t - 13$

$t = 5$

Distance travelled $= \int_0^5 |v(t)|\, dt$

$= \int_0^5 |6t^2 - 13t + 5|\, dt$

$= 115.7$ cm

determines *t* when *a* = 47 cm/s²✓

calculates distance travelled✓

Question 4 [SCSA MM2018 Q16] (8 marks)

(✓ = 1 mark)

a **i** By the fundamental theorem of calculus

$$\int_0^3 f'(x)\,dx = \left[f(x)\right]_0^3 = f(3) - f(0) = 2 - (-1) = 3$$

uses the fundamental theorem of calculus✓

obtains the correct value for the integral✓

ii By the fundamental theorem of calculus

$$\int_{-2}^3 f'(x)dx = \left[f(x)\right]_{-2}^3 = f(3) - f(-2) = 2 - 4 = -2$$

uses the fundamental theorem of calculus✓

obtains the correct value for the integral✓

b Required area is A.

$$A = \int_{-2}^{3} \left|f'(x)\right| dx$$

Since $f'(x)$ is positive for $x > 0$ and negative for $x < 0$, the area is

$$A = \int_{-2}^0 \left|f'(x)\right| dx + \int_0^3 \left|f'(x)\right| dx = \left|-1-4\right| + \left|2-(-1)\right| = 8$$

writes the expression for area in terms of absolute value✓

uses the intervals where $f'(x)$ is positive and negative✓

breaks the integral over the correct intervals✓

calculates the correct value of the area✓

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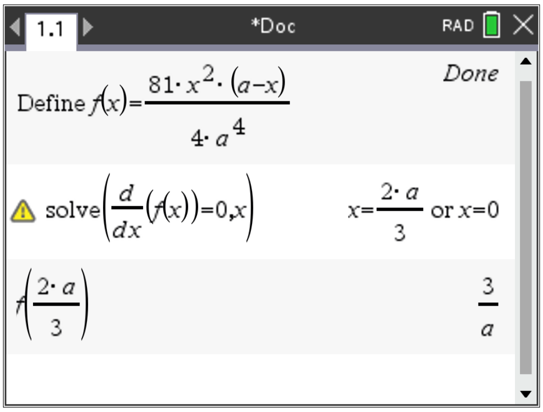
Question 5 (5 marks)

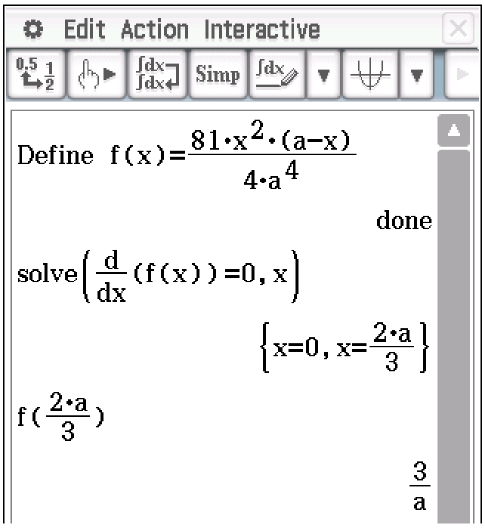
(✓ = 1 mark)

a Define $f(x)=\dfrac{81x^2(a-x)}{4a^4}$ for use for this question.

Set the derivative equal to 0 and solve for *x*.✓

Substitute the answer into $f(x)$ to find the *y*-coordinate.

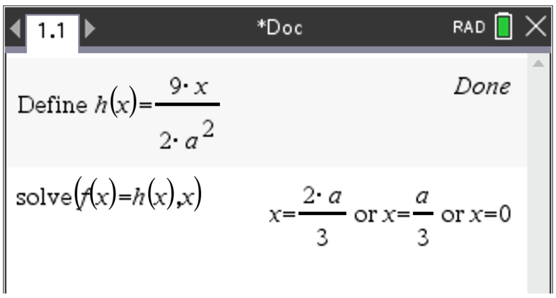


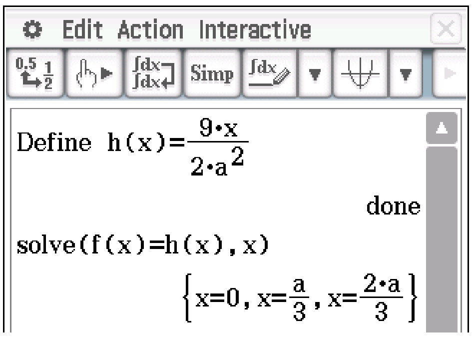


The coordinates of the local maximum are $\left(\dfrac{2a}{3},\dfrac{3}{a}\right)$.✓

b Define $h(x)=\dfrac{9x}{2a^2}$.

To determine the *x*-values of the points of intersection, solve $f(x)=h(x)$ for *x*.





The *x* values are $\boldsymbol{x=0}$, $\boldsymbol{x=\dfrac{a}{3}}$ and $\boldsymbol{x=\dfrac{2a}{3}}$.✓

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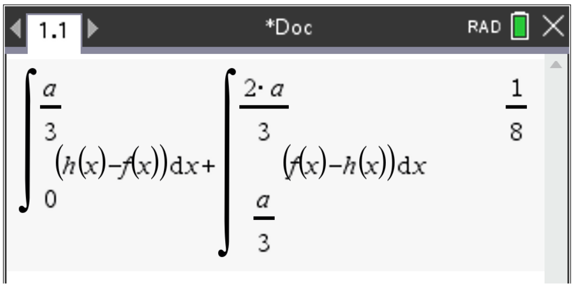
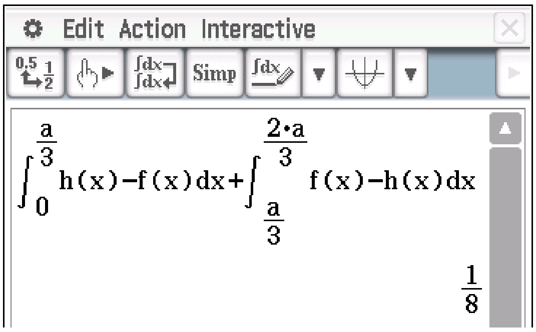
c As $a>0$, confirm by graphing that $h(x)$ is above $f(x)$ from $x=0$ to $x=\frac{a}{3}$ and $f(x)$ is above $h(x)$ from $x=\frac{a}{3}$ to $x=\frac{2a}{3}$. To find the area, calculate the sum of the integrals

$$\int_0^{\frac{a}{3}}\left(h(x)-f(x)\right)dx+\int_{\frac{a}{3}}^{\frac{2a}{3}}\left(f(x)-h(x)\right)dx\ .✓$$





The area between the curves is $\frac{1}{8}$ **units**^{2}.✓

Chapter 4 – Applying the exponential and trigonometric functions

EXERCISE 4.1 Exponential growth and decay

Question 1

a **i** N_0 is the initial number of people (found by letting $t = 0$)

Hence $\boldsymbol{N_0 = 200}$

ii $N(t) = 200e^{kt}$

$t = 1, N = 500$

$200e^k = 500$

$e^k = \frac{500}{200}$

$k = \log_e\left(\frac{500}{200}\right)$

$\boldsymbol{k = 0.9163}$

b $N(6) = 200e^{0.9163\times6}$

$= 48830.84035$

The number infected after 6 months is 48 831.

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Question 2

a **i** B_0 is the initial number of bacteria (found by letting $t = 0$)

Hence $\boldsymbol{B_0 = 100\,000}$

ii $B(t) = 100\,000e^{kt}$

$t = 5$, $N = 105\,000$

$105\,000 = 100\,000e^{5k}$

$$e^{5k} = \frac{105\,000}{100\,000}$$

$$k = \frac{1}{5}\log_e(1.05)$$

$\boldsymbol{k = 0.0098}$

b $t = 24$

$B(24) = 100\,000e^{24\times0.0098}$

$= \mathbf{126\,516}$

c $B(t) = 2 \times 100\,000 = 200\,000$

$200\,000 = 100\,000e^{0.0098t}$

$$t = \frac{1}{0.0098}\log_e(2)$$
$$\approx 70.73$$

It will take 71 hours for the bacteria to double.

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Question 3

a D_0 is the initial amount of ibuprofen in the body. This is 400 mg.

Hence $\boldsymbol{D_0 = 400}$

b $D(t) = 400e^{-kt}$

$t = 1,\ D = 280$

$280 = 400e^{-k}$

$$e^k = \frac{400}{280}$$

$$k = \log_e\left(\frac{400}{280}\right) = 0.35667...$$

$\boldsymbol{k = 0.3567}$

c $t = 2$

$D(2) = 400e^{-0.3567\times 2} = 816.367\ldots$

The amount after 2 hours is 816.4 mg.

Question 4

a $P(t) = 6191e^{0.04t}$

Year 1990 is $t = 0$.

Hence $P(0) = 6191e^0$

$\boldsymbol{= 6191}$

b Year 1991 is $t = 1$.

$P(1) = 6191e^{0.04\times 1}$

$= 6444$

The population in 1991 was 6444.

c $$\frac{P(1) - P(0)}{P(0)}\times 100 = \frac{6444 - 6191}{6191}\times 100$$

$\boldsymbol{\approx 4\%}$

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Question 5

a $t = 1,\ d = 10$, so $10 = d_0e^m$

$t = 2,\ d = 15$, so $15 = d_0e^{2m}$

The two equations are $\boldsymbol{d_0e^m = 10}$ **and** $\boldsymbol{d_0e^{2m} = 15}$**.**

b $\dfrac{d_0e^{2m}}{d_0e^m} = \dfrac{15}{10}$

$e^m = 1.5$

$m = \ln(1.5)$

$\boldsymbol{m = 0.405}$

$d_0e^{0.405} = 10$

$d_0 = 10\ e^{-0.405}$

$\boldsymbol{d_0 = 6.670}$

Question 6

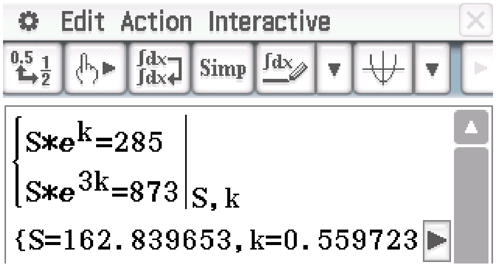
a $S_0 e^k = 285$

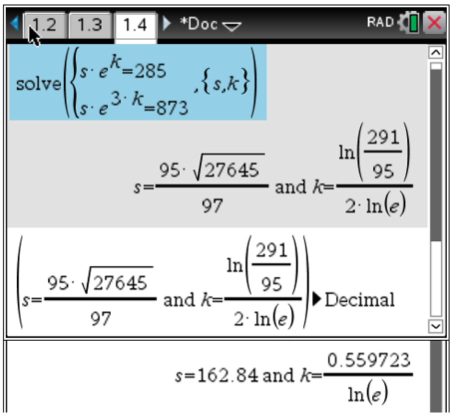
$$S_0 = \frac{285}{e^k} = \frac{285}{e^{0.5597}}$$

$$\approx \mathbf{163}$$

Alternatively, use CAS.

ClassPad **TI-Nspire**





b $S(t) = S_0 e^{kt}$

$t = 1$, $S = 285$ so $285 = S_0 e^k$

$t = 3$, $S = 873$ so $873 = S_0 e3^k$

$$\frac{873}{285} = \frac{S_0 e^{3k}}{S_0 e^k}$$

$$e^{2k} \approx 3.0632$$

$$2k \approx 1.1944$$

$$\boldsymbol{k = 0.5597}$$

Or use CAS

c $S(11) = 163e^{0.5597 \times 11} = 76\,915$

The number of mobile phones in 1995 was 76 915.

Question 7 (3 marks)

(✓ = 1 mark)

a Initial mass is 200 grams, so $\boldsymbol{P_0 = 200}$✓

b $P = 200e^{kt}$

When $t = 5$, $P = 200 \div 2 = 100$✓

$100 = 200e^{5k}$

$e^{5k} = \frac{1}{2}$

Hence $\boldsymbol{e^{-5k} = \frac{2}{1} = 2}$ ✓

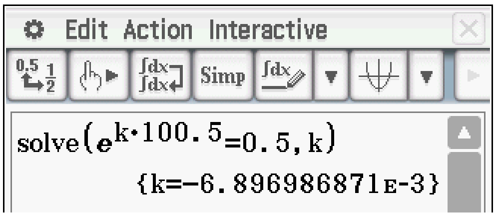
Question 8 [SCSA MM2016 Q9ab] (5 marks)

(✓ = 1 mark)

a $P = P_0 e^{kt}$

$e^{k \times 100.5} = \frac{1}{2}$

ClassPad



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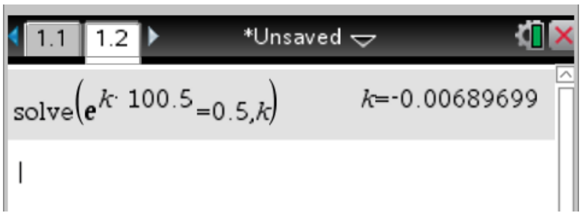

$k = -0.006896\ldots$

$k = -0.00690$

$k = -6.90 \times 10^{-3}$

sets up an equation to solve for *k*✓

solves for *k*✓

states *k* to three significant figures✓

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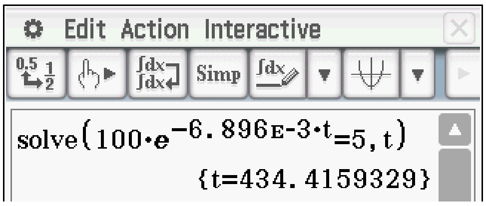
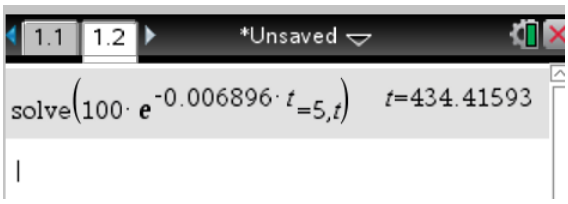
b **ClassPad**



TI N-Spire



$t = 434.42$ days

Accept 434 to 435 days.

sets up an equation to solve for *t*✓

determines the number of days✓

Question 9 [SCSA MM2018 Q9ab] (3 marks)

(✓ = 1 mark)

a Initial dose when $t = 0$

$C(0) = 4$ mg/L

determines concentration, including the unit✓

b $C = 4\ e^{-0.05(2.5)}$

$C = 3.53$ mg/L

substitutes t = 2.5✓

determines concentration, including the unit✓

Question 10 (5 marks)

(✓ = 1 mark)

a $N = N_0 e^{-kt}$

When $\boldsymbol{t = 10, N = \frac{1}{2}N_0}$ ✓

$$\frac{1}{2}N_0 = N_0 e^{-10k}$$

$$e^{-10k} = \frac{1}{2}$$

$$k = -\frac{1}{10}\log_e\left(\frac{1}{2}\right)$$

$$= 0.0693147...$$

$\boldsymbol{k = 0.0693}$ ✓

b Initial number of atoms is 8.0×10^{20}, so $\boldsymbol{N_0 = 8.0 \times 10^{20}}$ ✓

$$N = \left(8.0\times10^{20}\right)e^{-0.0693t}$$

$$t = 60, N = \left(8.0\times10^{20}\right)e^{-0.0693\times60}$$

$$\approx 1.25\times10^{19}$$

There are $\boldsymbol{1.25 \times 10^{19}}$ atoms remaining after 1 hour. ✓

c $N = N_0 e^{-kt}$

$$t = -\frac{1}{k}\log_e\left(\frac{N}{N_0}\right)$$

If there is to be no sample remaining, $N = 0$. However, $\log_e\left(\frac{N}{N_0}\right) = \log_e(0)$ is not possible since we need $\frac{N}{N_0} > 0$ or $N > 0$.

or

Alternatively, the graph has a horizontal asymptote at $N = 0$, therefore, it will never reach zero where there is no sample left.

Either explanation ✓

EXERCISE 4.2 Differentiating exponential functions

Question 1

a N_0 is the original number of people. It is found by using $t = 0$.

Hence $\mathbf{N_0 = 50}$.

b $N = 50e^{kt}$

$t = 1, N = 150$

$150 = 50e^{k}$

$k = \log_e\left(\frac{150}{50}\right)$

$\mathbf{k = 1.0986}$

Question 2

The decay is modelled by $N = N_0e^{-kt}$, where N_0 is the original number of atoms, N is the number of atoms present at time t and k is the decay rate of the material.

Here $N_0 = 1.6 \times 10^{21}$ and it takes 20 minutes for half the remaining atoms to decay, which means $t = 20$.

Hence, after the first 20 minutes the number of atoms gets reduced to half.

$N = 8.0 \times 10^{20}$

So, $8.0 \times 10^{20} = 1.6 \times 10^{21}(e^{-20k})$

$0.5 = e^{-20k}$

$\log_e(0.5) = -20k$

$k = -0.05\log_e(0.5)$

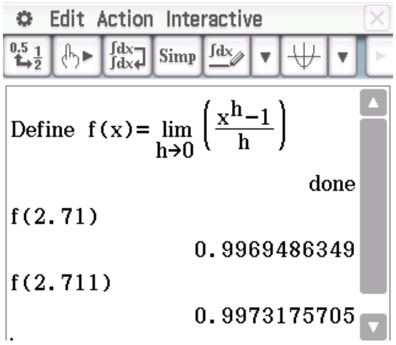
$= 0.034657\ldots$

Hence the value of k = **0.0347**.

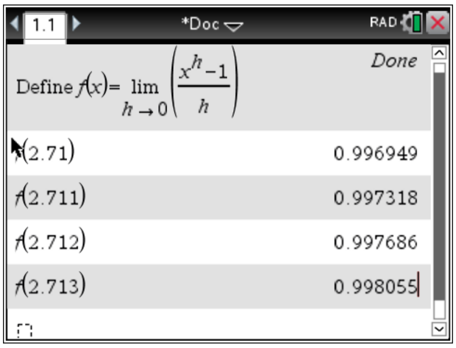
Question 3

Define the function and use it to produce as many iterations as needed to produce a value greater than 1.

ClassPad



TI-Nspire



a	$\lim\limits_{h \to 0} \frac{a^h - 1}{h}$
2.71	0.996 949
2.711	0.997 318
2.712	0.997 686
2.713	0.998 055
2.714	0.998 424
2.715	0.998 792
2.716	0.999 160
2.717	0.999 528
2.718	0.999 896
2.719	1.000 264
2.72	1.000 632

Choose the value of a which produces a limit closest to 1.

The approximation for the limit is 2.718.

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Question 4

a $y = 9e^x$

$$\frac{d}{dx}\left(9e^x\right) = 9\frac{d}{dx}\left(e^x\right) = \mathbf{9e^x}$$

b $y = e^x + x^2$

$$\frac{d}{dx}\left(e^x + x^2\right) = \frac{d}{dx}\left(e^x\right) + \frac{d}{dx}\left(x^2\right)$$

$$= \mathbf{e^x + 2x}$$

c $y = \left(2e^x - 3\right)^6$

Here we use the chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

Let $u = 2e^x - 3$

Then, $y = u^6$

So, $\frac{dy}{du} = 6u^5$

$$= 6\left(2e^x - 3\right)^5$$

$$\frac{du}{dx} = 2e^x$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(2e^x - 3\right)^6 = 6\left(2e^x - 3\right)^5 \times 2e^x$$

$$= \mathbf{12e^x\left(2e^x - 3\right)^5}$$

d $y = \frac{\left(e^x + e^{-x}\right)^2}{e^x}$

$$= e^{-x}\left(e^{2x} + 2 + e^{-2x}\right)$$

$$= e^x + 2e^{-x} + e^{-3x}$$

Hence,

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^x + 2e^{-x} + e^{-3x}\right)$$

$$= e^x - 2e^{-x} - 3e^{-3x}$$

$$= \mathbf{e^{-3x}\left(e^{4x} - 2e^{2x} - 3\right)}$$

e Here we use the chain rule $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$.

Let $u=2x-1$

Then, $y=e^u$

So, $\frac{dy}{du}=e^u$

$=e^{2x-1}$

$\frac{du}{dx}=2$

$\frac{dy}{dx}=\mathbf{2e^{2x-1}}$

f Here we use the chain rule.

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

$y=e^{\sqrt{2x+4}}$

Let, $u=\sqrt{2x+4}$

$y=e^u$

$\frac{dy}{du}=e^u=e^{\sqrt{2x+4}}$

$\frac{du}{dx}=\frac{1}{2}(2x+4)^{-\frac{1}{2}}\times 2$

$=(2x+4)^{-\frac{1}{2}}$

$\frac{dy}{dx}=(2x+4)^{-\frac{1}{2}}\times e^{\sqrt{2x+4}}$

$=\mathbf{\frac{e^{\sqrt{2x+4}}}{\sqrt{2x+4}}}$

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Question 5

a $f(x) = xe^x$

Use the product rule by identifying u and v and finding their derivatives with respect to x.

$f(x) = uv$

Using $\frac{d}{dx}(uv) = vu' + uv'$

$\boldsymbol{f'(x) = xe^x + e^x}$

b $f(x) = (2x+3)e^x$

Use the product rule by identifying u and v and finding their derivatives with respect to x.

$f(x) = uv$

Using $\frac{d}{dx}(uv) = vu' + uv'$

$f'(x) = (2x+3)e^x + e^x(2)$

$= \boldsymbol{2xe^x + 5e^x}$

c $f(x) = 5x^3e^x$

$f(x) = uv$

Use the product rule by identifying u and v and finding their derivatives with respect to x.

Using $\frac{d}{dx}(uv) = vu' + uv'$

$\boldsymbol{f'(x) = 5x^3e^x + 15x^2e^x}$

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Question 6

Let $w = \frac{u}{v} = \frac{e^x - 4}{\sqrt{e^x + 1}}$

$u = e^x - 4$ and $v = \sqrt{e^x + 1}$

$u' = e^x$ and $v' = \frac{1}{2}\left(e^x + 1\right)^{-\frac{1}{2}} e^x$

Using the quotient rule,

$$\frac{dw}{dx} = \frac{vu' - uv'}{v^2}$$

$$= \frac{\sqrt{e^x + 1} \times e^x - \frac{1}{2}\left(e^x + 1\right)^{-\frac{1}{2}} e^x \left(e^x - 4\right)}{e^x + 1}$$

$$g'(x) = \frac{e^x\sqrt{e^x + 1} - \frac{e^x\left(e^x - 4\right)}{2\sqrt{e^x + 1}}}{e^x + 1}$$

Now substitute $x = 3$.

$$\text{Then } g'(3) = \frac{e^3\sqrt{e^3 + 1} - \frac{e^3\left(e^3 - 4\right)}{2\sqrt{e^3 + 1}}}{e^3 + 1}$$

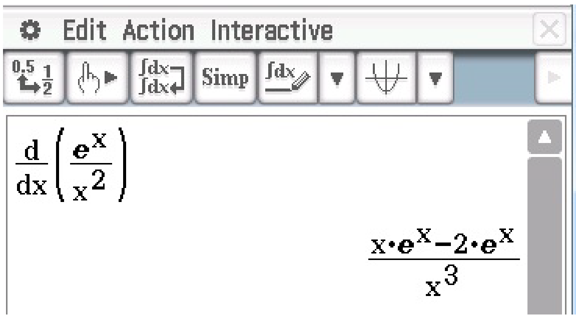
$$= \frac{e^3\left(e^3 + 6\right)}{2\sqrt{\left(e^3 + 1\right)^3}} = \frac{e^3\left(e^3 + 6\right)}{2\left(e^3 + 1\right)^{\frac{3}{2}}}$$

$= 2.7056\ldots$

$\approx$ **2.71**

Question 7

a **ClassPad**



1 In **Main**, enter and highlight the expression.

2 Tap **Interactive** > **Calculation** > **diff**.

3 In the dialogue box tap **Derivative**.

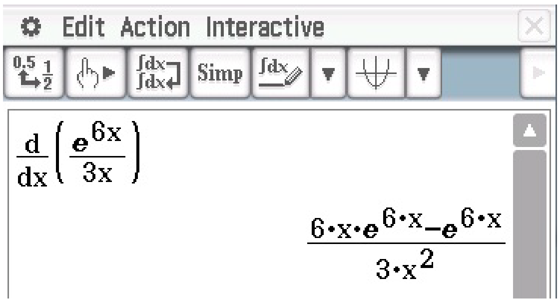
The derivative is $\dfrac{(x-2)e^x}{x^3}$.

TI-Nspire

1 Press **menu** > **Calculus** > **Derivative**.

2 In the template, enter the expression and press **enter**.

b **ClassPad**



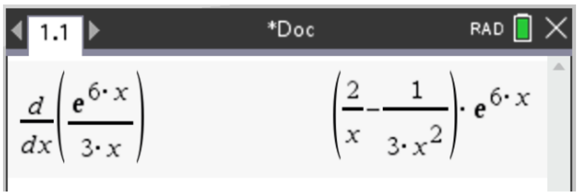
1 In **Main**, enter and highlight the expression.

2 Tap **Interactive** > **Calculation** > **diff**.

3 In the dialogue box tap **Derivative**.

The derivative is $\dfrac{(6x-1)e^{6x}}{3x^2}$.

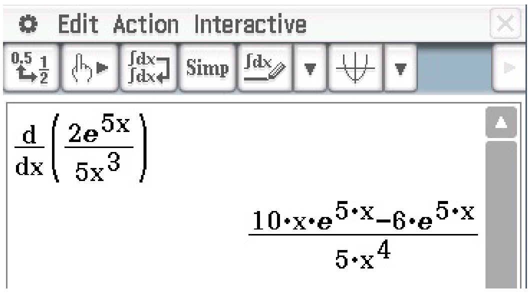
TI-Nspire



1 Press **menu** > **Calculus** > **Derivative**.

2 In the template, enter the expression and press **enter**.

c **ClassPad**



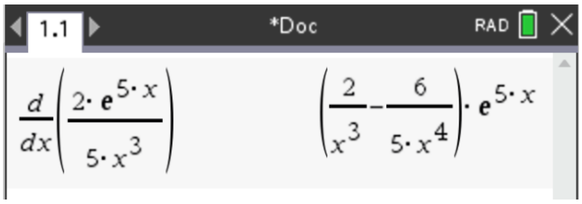
1 In **Main**, enter and highlight the expression.

2 Tap **Interactive** > **Calculation** > **diff**.

3 In the dialogue box tap **Derivative**.

The derivative is $\dfrac{2(5x-3)e^{5x}}{5x^4}$.

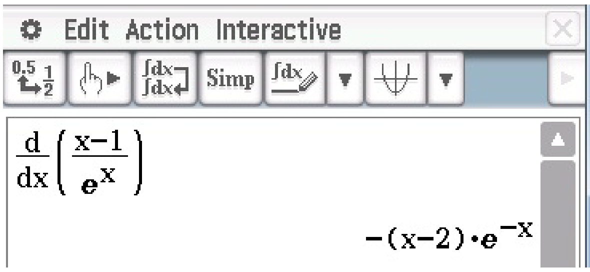
TI-Nspire



1 Press **menu** > **Calculus** > **Derivative**.

2 In the template, enter the expression and press **enter**.

d **ClassPad**



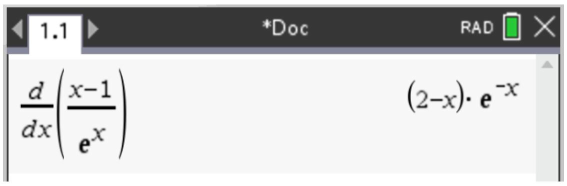
1 In **Main**, enter and highlight the expression.

2 Tap **Interactive** > **Calculation** > **diff**.

3 In the dialogue box tap **Derivative**.

The derivative is $\dfrac{2-x}{e^x}$.

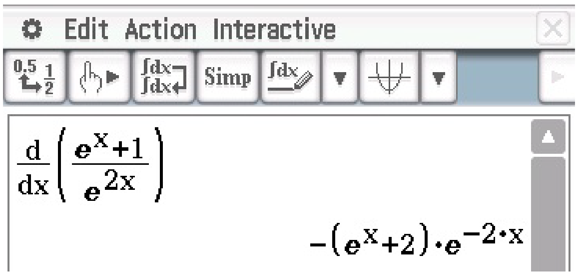
TI-Nspire



1 Press **menu** > **Calculus** > Derivative.

2 In the template, enter the expression and press **enter**.

e **ClassPad**



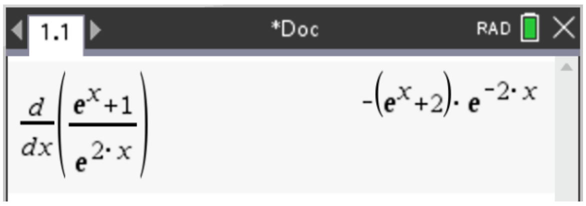
1 In **Main**, enter and highlight the expression.

2 Tap **Interactive > Calculation > diff**.

3 In the dialogue box tap **Derivative**.

The derivative is $-\dfrac{e^x+2}{e^{2x}}$.

TI-Nspire



1 Press **menu > Calculus > Derivative**.

2 In the template, enter the expression and press **enter**.

Question 8

$y = e^{4x}(x^3 - 3x + 5)$

$\quad = e^{4x} \times x^3 - 3xe^{4x} + 5e^{4x}$

Using the product rule, $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(e^{4x} \times x^3 - 3xe^{4x} + 5e^{4x}\right)$

$$= \frac{d}{dx}\left(e^{4x} \times x^3\right) - \frac{d}{dx}\left(3xe^{4x}\right) + \frac{d}{dx}\left(5e^{4x}\right)$$

$$= 3x^2e^{4x} + 4x^3e^{4x} - 3e^{4x} - 12xe^{4x} + 20e^{4x}$$

Substitute $x = -1$

Then $f(-1) = 3(-1)^2 e^{-4} + 4(-1)^3 e^{-4} - 3e^{-4} - 12(-1)e^{-4} + 20e^{-4}$

$$= 3e^{-4} - 4e^{-4} - 3e^{-4} + 12e^{-4} + 20e^{-4}$$

$$= 28e^{-4}$$

$$= \frac{28}{e^4}$$

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Question 9

$$f(x)=\frac{xe^{3x}+5}{x^2+e}$$

Using the quotient rule, let $u=xe^{3x}+5$, $v=x^2+e$ and then $u'=3xe^{3x}+e^{3x}$, $v'=2x$

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{vu'-uv'}{v^2}$$

$$f'(x)=\frac{\left(x^2+e\right)\left(3xe^{3x}+e^{3x}\right)-\left(xe^{3x}+5\right)\times 2x}{\left(x^2+e\right)^2}$$

Now substitute $x=2$ in $f'(x)$.

$$f'(2)=\frac{\left(2^2+e\right)\left(3\times 2e^{3\times 2}+e^{3\times 2}\right)-\left(2e^{3\times 2}+5\right)(2\times 2)}{\left(2^2+e\right)^2}$$

$$=\frac{(4+e)\left(6e^6+e^6\right)-8e^6-20}{(4+e)^2}$$

$$=\frac{(4+e)\left(7e^6\right)-8e^6-20}{(4+e)^2}$$

$$\approx 348.397$$

$$\approx \mathbf{348.4}$$

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Question 10

$h(x)=5x^2e^{3x}+e^x$

Using the product rule,

$$h'(x)=5x^2\times 3e^{3x}+5\times 2x\times e^{3x}+e^x$$
$$=15x^2e^{3x}+10xe^{3x}+e^x$$

Now substitute $x=2$.

$$h'(2)=15\times 2^2\times e^{3\times 2}+10\times 2\times e^{3\times 2}+e^2$$
$$=60e^6+20e^6+e^2$$
$$=\mathbf{80e^6+e^2}$$
$$=32\ 281.6925\ldots$$
$$\approx \mathbf{32\ 281.69}$$

Question 11

Let $f(x)=xe^{2x-1}$

Then, $f'(x)=x(2)e^{2x-1}+e^{2x-1}$

It is given in the question that the rate of change is equal to $5e^3$.

Then, $2xe^{2x-1}+e^{2x-1}=5e^3$

$$e^{2x-1}(2x+1)=5e^3$$

Equating coefficients we have $2x+1=5$.

Equating powers we have $2x-1=3$.

So, $\boldsymbol{x=2}$

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Question 12

$$f(x)=e^{\frac{x}{2}}, f(1)=\sqrt{e}$$

$$f'(x)=\frac{1}{2}e^{\frac{x}{2}}$$

$$m=f'(1)=\frac{1}{2}e^{\frac{1}{2}}=\frac{1}{2}\sqrt{e}$$

The equation of the tangent will be in the form $y-y_1=m(x-x_1)$.

At $(1,\sqrt{e})$,

$$y-\sqrt{e}=\frac{1}{2}\sqrt{e}(x-1)$$

$$y=\frac{1}{2}\sqrt{e}(x+1)$$

Hence the equation of the tangent is

$$\boldsymbol{y=\frac{1}{2}\sqrt{e}(x+1)=\frac{\sqrt{e}}{2}(x+1)}$$

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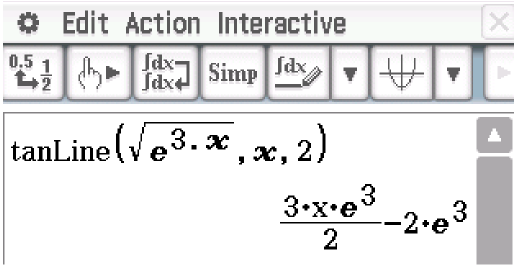
Question 13

ClassPad



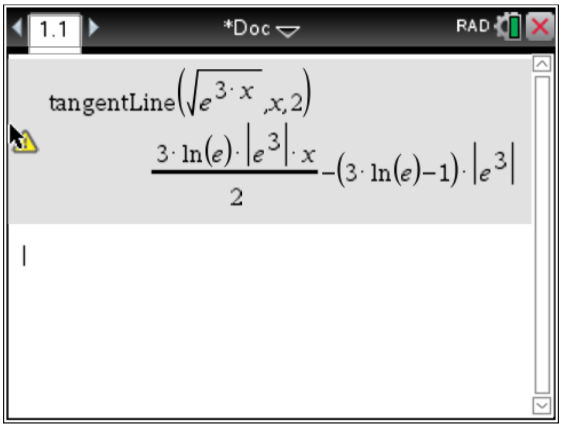
1 In Main, enter and highlight the expression $\sqrt{e^{3x}}$

2 Tap **Interactive** > **Calculation** > **line** > **tanLine**.

3 In the dialogue box, **Point:** field, enter **2**.

4 Tap **OK**.

TI-Nspire



1 Press **menu** > **Calculus** > **Tangent Line.**

2 Enter the expression $\sqrt{e^{3x}}$ followed by **, x, 2**.

3 Press **ctrl** + **enter** for the approximate solution.

4 Simplify the answer using $\ln(e)=1$ and $\left|e^3\right|=e^3$.

The equation of the tangent is $y=\dfrac{3e^3x}{2}-2e^3$ or $y=e^3\left(\dfrac{3x}{2}-2\right)$ or $y=\dfrac{e^3}{2}(3x-4)$

Question 14

a Use the chain rule, with $y = e^u$, where $u = 2x^2 - 4x$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= e^u(4x-4) \\ &= \mathbf{(4x-4)e^{2x^2-4x}}\end{aligned}$$

b Use the product rule and chain rule to differentiate $(4x-4)e^{2x^2-4x}$.

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}((4x-4))e^{2x^2-4x} + (4x-4)\frac{d}{dx}\left(e^{2x^2-4x}\right) \\ &= 4e^{2x^2-4x} + (4x-4)^2 e^{2x^2-4x} \\ &= (4 + 16x^2 - 32x + 16)e^{2x^2-4x} \\ &= \mathbf{(16x^2 - 32x + 20)e^{2x^2-4x}}\end{aligned}$$

c $\frac{dy}{dx} = (4x-4)e^{2x^2-4x}$

For a stationary point, $\frac{dy}{dx} = 0$.

$(4x-4)e^{2x^2-4x} = 0$

$4x - 4 = 0 \Rightarrow x = 1$

So $\boldsymbol{x = 1}$ is the x value of the stationary point.

d Work out the derivative value for an x value close to 1 on either side of $x = 1$.

For $x = 0.9$, $\frac{dy}{dx} < 0$, and for $x = 1.1$, $\frac{dy}{dx} > 0$.

The stationary point is a local minimum.

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Question 15

a Here the exponential model of this exponential growth is $N = 1100e^{0.025t}$.

Here 1100 is the initial value.

Hence, there were **1100** swans at the beginning of the study.

b To find the number of swans after 5 months, we substitute $t = 5$ in the exponential model.

$$N = 1100e^{0.025\times 5}$$
$$= 1100e^{0.125}$$
$$\approx 1246.46$$
$$= \mathbf{1246}$$

c To find the rate after 5 months, substitute $t = 5$ in the derivative.

$$\frac{dN}{dt} = 27.5e^{0.025t}$$

Then,

$$\frac{dN}{dt}(5) = 27.5e^{0.025\times 5}$$
$$= 31.1616...$$

So, the number of swans is increasing by 31 per month.

d

$$\delta N = \frac{dN}{dt}\delta t$$
$$= 27.5e^{0.025t}\delta t$$

$t = 5, \delta t = 0.1$

$$\delta N = 27.5e^{0.025\times 5} \times 0.1$$
$$= 3.11615\ldots$$
$$\approx \mathbf{3.1}$$

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Question 16

a The exponential model is given by $A = 120\,000\, e^{-0.033t}$.

To find the rate, find the derivative of A with respect to t.

$$\frac{dA}{dt} = 120\,000 \times (-0.033) \times e^{-0.033t}$$
$$= -3960e^{-0.033t}$$

To find the rate after 2 years, substitute $t = 2$.

$$\frac{dA}{dt} = -3960 \times e^{-0.033 \times 2}$$
$$\approx -3707.08$$

The rate of decrease is **3707 hectares per year.**

b To find the rate after 15 years, substitute $t = 15$.

$$\frac{dA}{dt} = -3960 \times e^{-0.033 \times 15}$$
$$\approx 2413.90$$

The rate of decrease is **2414 hectares per year.**

c To find the rate for 40 years, substitute $t = 40$.

$$\frac{dA}{dt} = -3960 \times e^{-0.033 \times 40}$$
$$\approx 1057.86$$

The rate of decrease is **1058 hectares per year.**

Question 17

a The initial amount occurs at $t = 0$. If the initial amount is 200 mg, so $200 = R_0 e^{k \times 0} = R_0$.

Hence $R_0 = 200$ mg.

b $R(t) = 200e^{kt}$

$t = 1$, $R = 191.6$

$191.6 = 200e^{k}$

$k = \log_e\left(\dfrac{191.6}{200}\right)$

$k = -0.043$

c Rate of decay $= R'(t) = R_0 \times k\, e^{kt} = 200 \times -0.043\, e^{kt} = -8.6e^{kt}$.

After 7 days, $R'(7) = -8.6\, e^{-0.043\,(7)}$

$= -6.3646688\ldots$

$\approx$ **-6.365 mg/day**

Question 18 (7 marks)

(✓ = 1 mark)

a **Differentiate $y = x^3e^{2x}$ with respect to x using the product rule.**✓

Identify u and $v: u = x^3, v = e^{2x}$.

Differentiate to obtain u' and v': $u' = 3x^2$, $v' = 2e^{2x}$

Write down the expression for the derivative of the product of functions and simplify.

$y'(x) = uv' + vu' = x^3 \times 2e^{2x} + e^{2x}3x^2$

$= 3x^2e^{2x} + 2x^3e^{2x}$

$= \mathbf{x^2e^{2x}(3+2x)}$ ✓

b $f(x) = e^{g(x)}$, then $\frac{df}{dx}(x) = g'(x)e^{g(x)}$

$\mathbf{f(x) = e^{x^2}}$ ✓

$\mathbf{f'(x) = 2xe^{x^2}}$ ✓

So, $\mathbf{f'(3) = 2 \times 3e^{3^2} = 6e^9}$ ✓

c $f(x) = e^{x^2-x+3}$

Use the chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Let $u = x^2 - x + 3$ $\quad \frac{du}{dx} = 2x - 1$

Let $y = e^u$ $\quad \frac{dy}{du} = e^u$

$\mathbf{f'(x) = \frac{dy}{dx} = e^u \times (2x-1)}$ ✓

$f'(x) = (2x-1)e^{x^2-x+3}$

$f'(1) = (2 \times 1 - 1)e^{1^2-1+3}$

$= 1 \times e^3$

$= \mathbf{e^3}$ ✓

Question 19 [SCSA MM2016 Q3] (4 marks)

(✓ = 1 mark)

a

$$f'(x)=\frac{2(x-1)e^x-e^x(x-1)^2}{e^{2x}}$$

$$=\frac{e^x(x-1)(2-x+1)}{e^{2x}}$$

$$=-\frac{(x-1)(x-3)}{e^x}$$

$$=\frac{-x^2+4x-3}{e^x}$$

uses quotient rule✓

simplifies expression✓

b

$$f'(x)=-\frac{(x-1)(x-3)}{e^x}$$

$$f'(1)=f'(3)=0$$

identifies stationary points as $f'(x)=0$ ✓

shows that this is true for $x=1, 3$✓

Question 20 (3 marks)

(✓ = 1 mark)

The function $f(x)=e^x+k$ has derivative $\boldsymbol{f'(x)=e^x}$.✓

The gradient of the tangent to f at $x=a$ is $\boldsymbol{f'(a)=e^a}$.✓

This tangent touches f at the point $(a, f(a))=(a, e^a+k)$.

The tangent line also passes through the point (0, 0).

Hence,

$$\frac{e^a+k-0}{a-0}=e^a$$

$$\Rightarrow e^a+k=ae^a$$

$$\Rightarrow \boldsymbol{k=e^a(a-1)}$$ ✓

Question 21 **[SCSA MM2020 Q15] (9 marks)**

(✓ = 1 mark)

a $T(0) = 200 - 175e^{-0.07(0)}$

$= 25^0 C$

states correct temperature✓

b $T(5) = 200 - 175e^{-0.07(5)}$

$= 76.68^0 C$

states correct temperature✓

c $100 = T_0 - 175e^{-0.07(5)}$

$T_0 = 100 + 175e^{-0.35}$

$\approx 223.32^0 C$

states correct equation to be solved✓

solves for T_0, giving changed temperature✓

d $T'(t) = 12.25e^{-0.07t}$

$T'(5) = 12.25e^{-0.07(5)}$

$= 8.63^0 C / \min$

states correct derivative of T with respect to *t*✓

calculates correct rate✓

e As time increases, the rate of change in the temperature of the water tends to 0.

The temperature of the water tends to the constant value of T_0.

states that the rate of change in the temperature tends to 0✓

states the water temperature approaches a constant✓

states the water temperature approaches T_0✓

Question 22 **[SCSA MM2018 Q14] (5 marks)**

(✓ = 1 mark)

a

h	$a = 2.60$	$a = 2.70$	$a = 2.72$	$a = 2.80$
0.1	1.002 65	**1.044 25**	1.052 41	1.084 49
0.001	0.955 97	0.993 75	**1.001 13**	**1.030 15**
0.000 01	0.955 52	**0.993 26**	**1.000 64**	1.029 62

correctly completes three table values✓

correctly completes all entries and rounds to 5 d.p.✓

b $a = e \approx 2.71828$

When $a = e$ the table shows that the value of $\frac{d}{dx}\left(a^x\right) = a^x \lim_{h\to 0}\left(\frac{a^h - 1}{h}\right)$.

It follows then from the definition that $\frac{d}{dx}\left(e^x\right) = e^x \times 1$

$$= e^x$$

states $a = e$ or 2.71828✓

explains table results✓

explains significance of table result for part b✓

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EXERCISE 4.3 Integrating exponential functions

Question 1

Use the chain rule to differentiate.

$u = x^2, \quad y = g(x) = e^u - 7$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= e^u \times 2x$

$= 2xe^{x^2}$

$g'(2) = 2 \times 2e^{2^2}$

$= \mathbf{4e^4}$

$= 218.3926\ldots$

$\approx \mathbf{218.393}$

Question 2

To find the gradient of the tangent at $x = 2$, first find the derivative of the function $y = e^{3x}$.

$y' = 3e^{3x}$

Now substitute $x = 2$ into $y' = 3e^{3x}$.

Thus $y' = 3e^6$ at $x = 2$.

Hence, the gradient of the tangent to the graph $y = e^{3x}$ at $x = 2$ is $\mathbf{3e^6}$.

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Question 3

a Use $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$ with $a = -2$, $b = 0$.

$\int e^{-2x}\,dx = \boldsymbol{-\frac{1}{2}e^{-2x} + c}$

b Use $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$ with $a = 4$, $b = 0$.

$\int 5e^{4x}\,dx = \frac{1}{4} \times 5e^{4x} + c$

Hence, $\int 5e^{4x}\,dx = \boldsymbol{\frac{5}{4}e^{4x} + c}$

c Use $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$ with $a = 2$, $b = 1$.

$\int e^{2x+1}\,dx = \boldsymbol{\frac{1}{2}e^{2x+1} + c}$

d First, we separate the terms as
$\int (3e^{-2x} + e^{4x})\,dx = \int 3e^{-2x}\,dx + \int e^{4x}\,dx$

Use $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$ with $a = -2$, $b = 0$, for the first integral and $a = 4$, $b = 0$ for the second integral.

$\int 3e^{-2x}\,dx = -\frac{3}{2}e^{-2x} + c$

and

$\int e^{4x}\,dx = \frac{1}{4}e^{4x} + c$

Hence, $\int (3e^{-2x} + e^{4x})\,dx = \boldsymbol{-\frac{3}{2}e^{-2x} + \frac{1}{4}e^{4x} + c}$

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e First, we separate the terms as

$$\int \frac{e^{4x}-1}{e^x}\mathrm{d}x = \int \frac{e^{4x}}{e^x}\mathrm{d}x - \int \frac{1}{e^x}\mathrm{d}x$$

On simplification, we get

$$\int \frac{e^{4x}-1}{e^x}\mathrm{d}x = \int e^{3x}dx - \int e^{-x}dx$$

Now we use $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$ with $a = 3$, $b = 0$ for the first integral and $a = -1$, $b = 0$ for the second integral.

$$\int e^{3x}\,dx - \int e^{-x}\,dx = \frac{1}{3}e^{3x} - (-e^{-x}) + c$$

$$= \boldsymbol{\frac{1}{3}e^{3x} + e^{-x} + c}$$

f First expand the brackets and simplify the expression.

$$(e^{3x} - e^{-3x})^2 = e^{6x} - 2e^{3x}e^{-3x} + e^{-6x}$$

$$= e^{6x} - 2 + e^{-6x}$$

$$\int (e^{3x} - e^{-3x})^2\,dx = \int (e^{6x} - 2 + e^{-6x})\,dx$$

$$= \int e^{6x}\,dx - 2\int dx + \int e^{-6x}\,dx$$

$$= \boldsymbol{\frac{1}{6}e^{6x} - 2x - \frac{1}{6}e^{-6x} + c}$$

Question 4

Integrate $f'(x)$ to find $f(x)$.

Let $f'(x) = 2e^{4x}$

$\int f'(x)\,dx = \int 2e^{4x}\,dx$

Use $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$ with $a = 4$, $b = 0$.

$$\begin{aligned}\int 2e^{4x}\,dx &= 2\int e^{4x}\,dx \\ &= \frac{2}{4}e^{4x} + c \\ &= \frac{1}{2}e^{4x} + c\end{aligned}$$

Now to find the equation of the curve, we need to find the value of c.

Substitute $x = 0$, $y = 2$ in the equation $y = \frac{1}{2}e^{4x} + c$.

$y = \frac{1}{2}e^{4x} + c$

$2 = \frac{1}{2}e^{0} + c$

$2 = \frac{1}{2} + c$

$c = \frac{3}{2}$

Hence, the equation of the curve is

$\boldsymbol{y = \frac{1}{2}e^{4x} + \frac{3}{2}}$.

Question 5

Integrate $f'(x)$ to find $f(x)$.

Let $f'(x) = \frac{e^{3x} - 1}{e^x}$.

$\int f'(x)\,dx = \int \frac{e^{3x} - 1}{e^x}\,dx$

First, we separate the integral.

$\int \frac{e^{3x} - 1}{e^x}\,\text{dx} = \int \frac{e^{3x}}{e^x}\,\text{dx} - \int \frac{1}{e^x}\,\text{dx}$

On simplification, we get

$\int \frac{e^{3x} - 1}{e^x}\,\text{dx} = \int e^{2x}\,dx - \int e^{-x}\,dx$.

Now we use $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$ with $a = 2$, $b = 0$, for the first integral and $a = -1$, $b = 0$ for the second integral.

$$\int e^{2x}\,dx - \int e^{-x}\,dx = \frac{1}{2}e^{2x} - (-e^{-x}) + c$$

$$= \frac{1}{2}e^{2x} + e^{-x} + c$$

Now to find the equation of the curve, we need to find the value of c.

Substitute $x = 0$, $y = 11$ in the equation $y = \frac{1}{2}e^{2x} + e^{-x} + c$.

$$11 = \frac{1}{2}e^{0} + e^{0} + c$$

$$11 = \frac{1}{2} + 1 + c$$

$$11 = \frac{3}{2} + c$$

$$c = \frac{19}{2}$$

Hence, the equation of the curve is

$$\boldsymbol{y = \frac{1}{2}e^{2x} + e^{-x} + \frac{19}{2}}.$$

Question 6

Integrate $f'(x)$ to find $f(x)$.

$f'(x) = 5e^{-2x}$

$\int f'(x)\,dx = \int 5e^{-2x}\,dx$

Now we use $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$ with $a = -2$, $b = 0$.

$\int 5e^{-2x}\,dx = -\frac{5}{2}e^{-2x} + c$

Now to find the equation of the curve, we need to find the value of c.

We have $f(3) = 2e$, that is, when $x = 3$, $y = 2e$.

Substitute $x = 3$ and $y = 2e$ into the equation and solve for c.

$f(x) = -\frac{5}{2}e^{-2x} + c$

$2e = -\frac{5}{2}e^{-6} + c$

$2e + \frac{5e^{-6}}{2} = c$

$c = \frac{5}{2e^6} + 2e$

$= \frac{4e^7 + 5}{2e^6}$

Hence, the function is

$$\boldsymbol{f(x) = -\frac{5}{2}e^{-2x} + \frac{4e^7 + 5}{2e^6}}$$

Question 7

a $\int_0^4 e^x \mathrm{d}x = \left[e^x\right]_0^4$

$= \boldsymbol{e^4 - 1}$

b $\int_1^3 5e^x \mathrm{d}x = 5\left[e^x\right]_1^3$

$= 5(e^3 - e)$

$= \boldsymbol{5e^3 - 5e}$

c Integrate each term separately.

$\int_2^4 \left(x^3 - e^x\right) \mathrm{d}x = \int_2^4 x^3 \mathrm{d}x - \int_2^4 e^x \mathrm{d}x$

$= \left[\frac{x^4}{4}\right]_2^4 - \left[e^x\right]_2^4$

$= \frac{1}{4}\left(4^4 - 2^4\right) - \left(e^4 - e^2\right)$

$= \frac{1}{4}(240) - e^4 + e^2$

$= \boldsymbol{60 - e^4 + e^2}$

$= \boldsymbol{-e^4 + e^2 + 60}$

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Question 8

a Integrate each term separately.

$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\left(\sin(3x)+e^{-6x}\right)\mathrm{d}x=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\sin(3x)\,\mathrm{d}x+\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}e^{-6x}\,\mathrm{d}x\ \mathrm{T}$$

$$=-\frac{1}{3}\left[\cos 3x\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}-\frac{1}{6}\left[e^{-6x}\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$

$$=0-\frac{1}{6}\left(e^{-\pi}-e^{\pi}\right)$$

$$=\frac{1}{6}\left(e^{\pi}-e^{-\pi}\right)$$

$$\approx \mathbf{3.85}$$

b Integrate each term separately.

$$\int_0^{\pi}\left(\cos\left(\frac{x}{2}\right)+e^{3x}\right)\mathrm{d}x=\int_0^{\pi}\cos\left(\frac{x}{2}\right)\mathrm{d}x+\int_0^{\pi}e^{3x}\,\mathrm{d}x$$

$$=2\left[\sin\left(\frac{x}{2}\right)\right]_0^{\pi}+\frac{1}{3}\left[e^{3x}\right]_0^{\pi}$$

$$=2(1-0)+\frac{1}{3}\left(e^{3\pi}-1\right)$$

$$=2+\frac{1}{3}\left(e^{3\pi}\right)-\frac{1}{3}$$

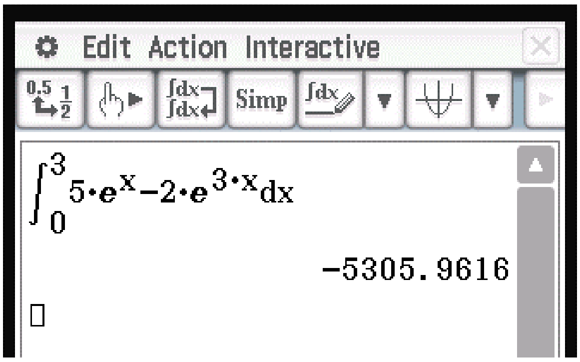
$$=\frac{1}{3}\left(5+e^{3\pi}\right)$$

$$=4132.2159\ldots$$

$$\approx \mathbf{4132.22}$$

Question 9

Casio



1 In **Main**, enter and highlight the expression.

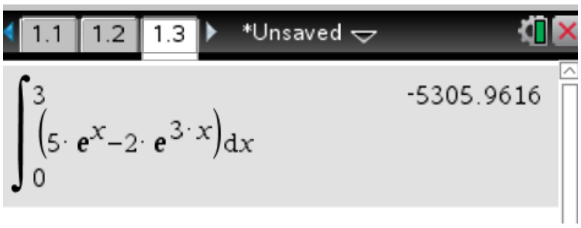
2 Tap **Interactive** > **Calculation** > ∫.

3 In the dialogue box tap **Definite** to enter the lower and upper limits.

4 Tap **OK**.

The answer is –**5306.0** to 1 decimal place.

TI-Nspire



1 Press **menu** > **Calculus** > **Integral**.

2 Enter the lower limit, the upper limit and the expression.

3 Press **ctrl** + **enter** for the approximate solution.

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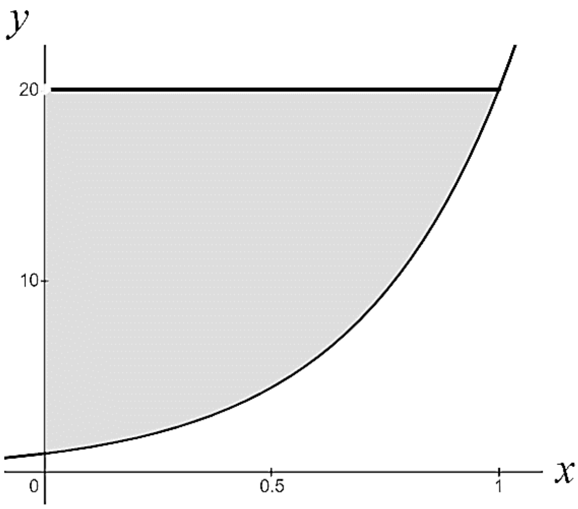
Question 10



Find the point of intersection of the straight line $y = e^3$ and the curve $y = e^{3x}$.

$e^3 = e^{3x}$, so $x = 1$.

The shaded area is found using the integral

$$\int_0^1 \left(e^3 - e^{3x}\right) dx = \left[e^3 x - \frac{1}{3}e^{3x}\right]_0^1$$

$$= \left(e^3 - \frac{1}{3}e^3\right) - \left(0 - \frac{1}{3}\right)$$

$$= \mathbf{\frac{2}{3}e^3 + \frac{1}{3} = \frac{2e^3}{3} + \frac{1}{3}\ \text{unit}^2}$$

Question 11 **[SCSA MM2016 Q2] (5 marks)**

(✓ = 1 mark)

a $\frac{d}{dx}\left(2xe^{2x}\right) = 2x\left(2e^{2x}\right) + e^{2x}(2)$

$= 2(2x+1)e^{2x}$

uses product rule✓

differentiates exponential term✓

b $\frac{d}{dx}\left(2xe^{2x}\right) = 2x\left(2e^{2x}\right) + e^{2x}(2)$

$\int \frac{d}{dx}\left(2xe^{2x}\right)dx = \int 2x\left(2e^{2x}\right)dx + \int 2e^{2x}dx$

$2xe^{2x} = \int 4xe^{2x}dx + e^{2x}$

$\int 4xe^{2x}dx = (2x-1)e^{2x} + c$

uses linearity of anti-differentiation✓

uses fundamental theorem✓

obtains an expression for required integral with a constant✓

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Question 12 **[SCSA MM2016 Q6] (4 marks)**

(✓ = 1 mark)

$$A = \int_0^4 \left(\left(6 - 2e^{x-4}\right) - \left(-\frac{1}{4}x + 5\right) \right) dx$$

$$= \int_0^4 \left(-2e^{x-4} + \frac{1}{4}x + 1 \right) dx$$

$$= \left[-2e^{x-4} + \frac{x^2}{8} + x \right]_0^4$$

$$= \left(-2 + 2 + 4\right) - \left(-2e^{-4}\right)$$

$$= 2\left(2 + \frac{1}{e^4} \right)$$

sets up an appropriate integral for area✓

uses correct limits✓

anti-differentiates correctly✓

calculates area✓

Question 13 (5 marks)

(✓ = 1 mark)

a

The graph of the function $f(x)=e^{\frac{x}{2}}+1$ crosses the y-axis $x=0$.

Substitute $x=0$, to find the y-coordinate.

$f(0)=e^{0}+1=2.$

Hence $f(0) = 2$.

Let m_1 be the gradient of the tangent to $f(x)$.

$f'(x)=\frac{e^{\frac{x}{2}}}{2} \Rightarrow m_1 = f'(0)=\frac{1}{2}.$

Use the point-gradient formula to find the equation of $f(x)$.

$y-2=f'(0)(x-0)$

$y-2=\frac{1}{2}x$

$y=\frac{x}{2}+2.$

This is the equation of the tangent passing through (0, 2).

The gradient of the normal is $m_2=\frac{-1}{m_1}=\frac{-1}{\frac{1}{2}}=-2.$

Use the point-gradient formula to find the equation of the normal.

$y-2=m_2(x-0)$

$y-2=-2x$

$y=-2x+2$

So $y=-2x+2$ is the equation of the normal to the graph of $f(x)$ where it crosses the y-axis.

correct calculation of gradient✓

correct calculation of y-intercept✓

b The x-coordinate of the point where the normal crosses the x-axis is found by substituting $y = 0$ into the equation of the normal.

$0 = -2x + 2$

$\boldsymbol{x = 1}$ ✓

The region is bounded above by $f(x) = e^{\frac{x}{2}} + 1$ and below by $y = -2x + 2$ from $x = 0$ to $x = 1$.

The area is given by the integral,

$$\boldsymbol{A = \int_0^1 \left(e^{\frac{x}{2}} + 1\right) - (-2x + 2)\,dx}$$ ✓

$$= \int_0^1 \left(e^{\frac{x}{2}} + 2x - 1\right) dx$$

$$= \left[\frac{e^{\frac{x}{2}}}{\frac{1}{2}} + 2\frac{x^2}{2} - x\right]_0^1$$

$$= \left(2e^{\frac{1}{2}} + 1 - 1\right) - \left(2e^0 + 0 - 0\right)$$

$A = 2\sqrt{e} - 2.$

Thus, the area of the shaded region is $A = \boldsymbol{2\sqrt{e} - 2}$ or $A = \boldsymbol{2e^{\frac{1}{2}} - 2}$. ✓

Question 14 (7 marks)

(✓ = 1 mark)

a

The function is $f(x) = 2e^{-\frac{x}{5}}$

Point P has coordinates $\left(x, 2e^{-\frac{x}{5}}\right)$, which indicates that the triangle has a base of length x and a height of $f(x)$.

The area of the right-angled triangle OQP is $A = \frac{1}{2}$ base × height

$$= \frac{1}{2} x \times 2e^{-\frac{x}{5}}$$

$$= \boldsymbol{xe^{-\frac{x}{5}}} \textbf{ units}^2 ✓$$

b

Given the formula for the area of the rectangle $A(x) = xe^{-\frac{x}{5}}$ a local maximum may occur when $\boldsymbol{A'(x) = 0}$.✓

Calculate this derivative using the product rule:

$$\frac{dA}{dx} = e^{-\frac{x}{5}} + x \times \left(-\frac{1}{5}\right)e^{-\frac{x}{5}} = e^{-\frac{x}{5}} - \frac{1}{5}xe^{-\frac{x}{5}} = e^{-\frac{x}{5}}\left(1 - \frac{x}{5}\right) ✓$$

To determine where $\frac{dA}{dx} = 0$ solve for x:

$$e^{-\frac{x}{5}}\left(1 - \frac{x}{5}\right) = 0.$$

As $e^{-\frac{x}{5}} \neq 0$, the equation requires $1 - \frac{x}{5} = 0$,

So, $x = 5$.

A graph of $A(x)$ confirms that the stationary point at $x = 5$ is a local maximum.

The maximum area of the triangle is then found by substituting $x = 5$ into $A(x)$.

$A(5) = 5e^{-\frac{5}{5}} = \boldsymbol{5e^{-1}} = \boldsymbol{\frac{5}{e}}$ units^2.✓

c The x value of the point T *satisfies* $f(x)=2e^{-\frac{x}{5}}=\frac{1}{2}$

$\Rightarrow e^{-\frac{x}{5}}=\frac{1}{4}$

$\Rightarrow -\frac{x}{5}=\log_e\left(\frac{1}{4}\right)$

$\therefore \boldsymbol{x=-5\log_e\left(\frac{1}{4}\right)}$ or $\boldsymbol{x=5\log_e(4)}$ ✓

Point S has coordinates $(0, f(0))=(0, 2)$.

Area shaded = Area of trapezium under ST – area enclosed by f, the x-axis and $x=0$ and $x=5\log_e(4)$.

$\boldsymbol{A=\frac{\frac{1}{2}+2}{2}\times 5\log_e(4)-\int_0^{5\log(4)}2e^{-\frac{x}{5}}dx}$ ✓

$=\frac{25}{4}\log_e(4)-\left[-10e^{-\frac{x}{5}}\right]_0^{5\log_e(4)}$

$=\frac{25}{4}\log_e(4)+10\left(\frac{1}{4}-1\right)$

$\boldsymbol{=\frac{25}{4}\log_e(4)-\frac{15}{2}}$ **units^2** ✓

= 1.16 units^2

Question 15 **[SCSA MM2017 Q15] (10 marks)**

(✓ = 1 mark)

a

The x value of the point T *satisfies* $V'(h) = e^{\left(-\frac{h^2}{100}\right)}$

So $V'(0.5) = e^{-0.0025} = 0.9975\, m^3/m$

uses Fundamental Theorem Calculus✓

obtains correct value for the rate of change✓

b It means the rate of change of the volume with respect to height when the height has reached 0.5 metres.

states meaning✓

c **i** Now $h(t) = 3t^2 - t + 4 = 6 \Rightarrow 3t^2 - t - 2 = 0 \Rightarrow (3t+2)(t-1) = 0$

So $t = 1$.

Then,

$$\frac{dh}{dt} = 6t - 1$$

$$\left.\frac{dh}{dt}\right|_{t=1} = 6(1) - 1 = 5 \text{ m/s}$$

differentiates *h* with respect to *t* correctly✓

state equation for time and substitutes values correctly✓

ii

$$\begin{aligned}\frac{dV}{dt} &= \frac{dV}{dh} \times \frac{dh}{dt} \\ &= e^{-\frac{6^2}{100}} \times 5 \\ &\approx 3.49 \text{ m}^3/\text{s}\end{aligned}$$

demonstrates use of the chain rule✓

substitutes values correctly to determine rate of change✓

iii $h(2)=3(2)^2-2+4=14$

$$\frac{\delta V}{\delta t}\approx\frac{dV}{dt}$$

$$\begin{aligned}\delta V&\approx e^{-\frac{14^2}{100}}\times 11\times\delta t\\&\approx 1.54944\times 0.1\\&\approx 0.155\end{aligned}$$

$$\begin{aligned}V(t=2.1)&\approx 8.439+0.155\\&\approx 8.594\text{ m}^3\end{aligned}$$

determines $h(2)$✓

uses incremental formula to approximate dV✓

estimates new V✓

EXERCISE 4.4 Differentiating trigonometric functions

Question 1

$$\int\left(e^{2x}-e^{-2x}\right)^2 dx=\int\left(e^{4x}+e^{-4x}-2\right)dx$$

$$=\frac{1}{4}e^{4x}-\frac{1}{4}e^{-4x}-2x+c$$

$$=\frac{e^{4x}}{4}-2x-\frac{e^{-4x}}{4}+c$$

Question 2

$$\int_0^3 4e^{2x}dx=\left[2e^{2x}\right]_0^3$$

$$=2e^6-2e^0$$

$$=\mathbf{2e^6-2}$$

$$=\mathbf{2\left(e^6-1\right)}$$

Question 3

Differentiate the function using the product rule, $\frac{d}{dx}(uv)=uv'+vu'$.

$u=x^2,\quad v=\sin(x)$

$u'=2x,\quad v'=\cos(x)$

$uv'+vu'=x^2\cos(x)+2x\sin(x)$

$$\mathbf{f'(x)=x^2\cos(x)+2x\sin(x)}$$

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Question 4

$$y=\frac{5\cos(3x)}{\sin(2x)}$$

Use the quotient rule: let $u=5\cos(3x)$ and $v=\sin(2x)$.

Differentiate to obtain u' and v'.

$u'=-15\sin(3x)$, $v'=2\cos(2x)$

Write the expression for $\frac{vu'-uv'}{v^2}$ and simplify.

$$\begin{aligned}\frac{dy}{dx}&=\frac{vu'-uv'}{v^2}\\&=\frac{\sin(2x)(-15\sin(3x))-5\cos(3x)(2\cos(2x))}{\sin^2(2x)}\\&=\frac{-15\sin(2x)\sin(3x)-10\cos(2x)\cos(3x)}{\sin^2(2x)}\\&=-\boldsymbol{\frac{15\sin(2x)\sin(3x)+10\cos(2x)\cos(3x)}{\sin^2(2x)}}\end{aligned}$$

Question 5

Use the chain rule.

$$u=4x^5-x,\quad \frac{du}{dx}=20x^4-1$$

$$y=\sin(u),\quad \frac{dy}{du}=\cos(u)$$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

$$\frac{dy}{dx}=\cos(u)\times(20x^4-1)$$

$$\boldsymbol{\frac{dy}{dx}=(20x^4-1)\cos(4x^5-x)}$$

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Question 6

$y = 4x^2 \cos\left(x^3\right)$

The product rule and chain rule are required to obtain the derivative.

Let $u = 4x^2$ and $v = \cos\left(x^3\right)$ in the product rule.

Differentiate to obtain u' and v'.

$u' = 8x$

Let $v = \cos\left(t\right)$ with $t = x^3$.

Use the chain rule $\frac{dv}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$ to find v'.

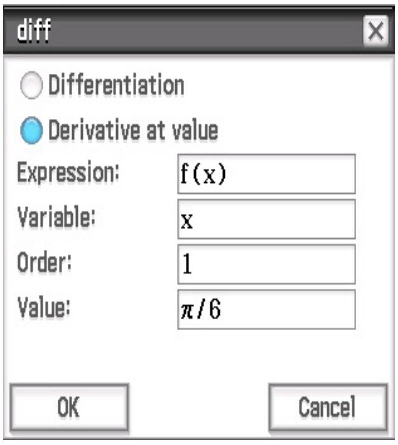
$$\begin{aligned}\frac{dv}{dx} &= \frac{dv}{dt} \times \frac{dt}{dx} \\ &= -\sin\left(t\right)\left(3x^2\right) \\ &= -3x^2 \sin\left(x^3\right)\end{aligned}$$

Use the product rule $\frac{dy}{dx} = uv' + vu'$.

$$\begin{aligned}\frac{dy}{dx} &= 4x^2\left(-3x^2 \sin(x^3)\right) + \cos x^3 \left(8x\right) \\ &= \mathbf{-12x^4 \sin\left(x^3\right) + 8x\cos\left(x^3\right)}\end{aligned}$$

Question 7

a **ClassPad**

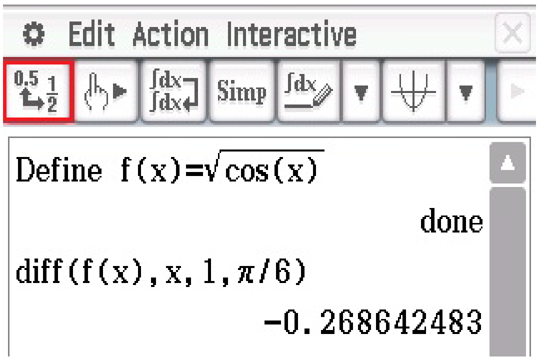


1 Define f(x) = $\sqrt{\cos(x)}$

2 Highlight f(x) and tap **Interactive > Calculation > diff**.

3 In the dialogue box, tap **Derivative at value**.

4 Enter Value: $\frac{\pi}{6}$



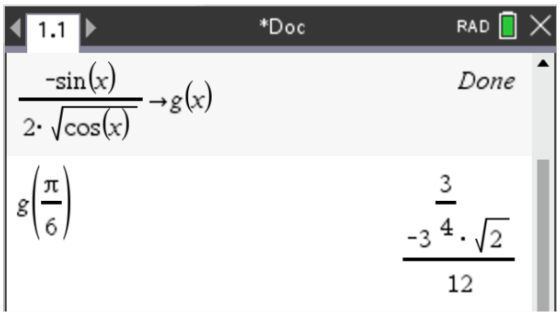
5 Tap **OK** and answer will be shown.

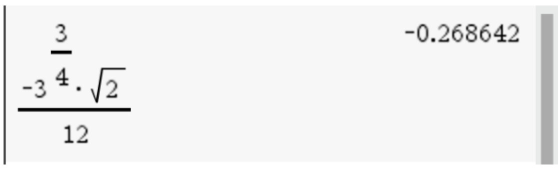
6 Change to decimals if required using **Convert** or by tapping **Decimal** at the bottom of the screen.

TI-Nspire

1 Define **f(x)** as shown above.

2 Find the derivative of **f(x)**.





3 Press **ctrl + var** to store the derivative as **g(x)**.

4 Enter $g\left(\frac{\pi}{6}\right)$ and press **ctrl + enter** for the approximate answer.

$$f'\left(\frac{\pi}{6}\right) = -0.269$$

b

Given $f'(x) = -\frac{\sqrt{6}}{4}$, show that cos($x$) satisfies the equation $2c^2 + 3c - 2 = 0$ and hence find the value of x if $0 < x < \pi$.>

$$\frac{-\sin(x)}{2\sqrt{\cos(x)}} = -\frac{\sqrt{6}}{4}$$

$$2\sin(x) = \sqrt{6\cos(x)}$$

$$4\sin^2(x) = 6\cos(x)$$

$$2\sin^2(x) = 3\cos(x)$$

$$2\left[1-\cos^2(x)\right] = 3\cos(x)$$

$$2 - 2\cos^2(x) = 3\cos(x)$$

$$0 = 2\cos^2(x) + 3\cos(x) - 2$$

If $c = \cos(x)$, then the above equation becomes $2c^2 + 3c - 2 = 0$

Solve for c to get:

$(2c - 1)(c + 2) = 0$

$c = \frac{1}{2}$ or -2

$\cos(x) = \frac{1}{2}$ or -2 (not possible since minimum possible value is -1)

$x = \frac{\pi}{3}$

$\boldsymbol{x = 1.047}$

Question 8

a The velocity can be found by $\text{velocity} = \frac{dx}{dt}$.

$$x = 3\cos\left(\frac{t}{2}\right)$$

$$v = 3\times\left(-\frac{1}{2}\sin\left(\frac{t}{2}\right)\right)$$

$$= \mathbf{-\frac{3}{2}\sin\left(\frac{t}{2}\right)}$$

b The acceleration a can be obtained by differentiating the velocity equation.

$$v = -\frac{3}{2}\sin\left(\frac{t}{2}\right)$$

$$a = -\frac{3}{2}\times\frac{1}{2}\cos\left(\frac{t}{2}\right)$$

$$= \mathbf{-\frac{3}{4}\cos\left(\frac{t}{2}\right)}$$

c Substitute $x = 0$ into $x = 3\cos\left(\frac{t}{2}\right)$ and solve for t.

$$0 = 3\cos\left(\frac{t}{2}\right)$$

$$\cos\left(\frac{t}{2}\right) = 0$$

It follows that $\frac{t}{2} = \frac{\pi}{2}$ or $\frac{t}{2} = \frac{3\pi}{2}$, that is, $t = \pi, 3\pi$.

The particle is at $x = 0$ when $\boldsymbol{t = \pi}$ **s** and $\boldsymbol{t = 3\pi}$ **s.**

d When $t = \pi$,

$$v = -\frac{3}{2}\sin\left(\frac{\pi}{2}\right)$$
$$= -\frac{3}{2} \times 1$$
$$= -\frac{3}{2}$$

and

$$a = -\frac{3}{4}\cos\left(\frac{\pi}{2}\right)$$
$$= -\frac{3}{4} \times 0$$
$$= 0$$

When $t = 3\pi$,

$$v = -\frac{3}{2}\sin\left(\frac{3\pi}{2}\right)$$
$$= -\frac{3}{2} \times (-1)$$
$$= \frac{3}{2}$$

and

$$a = -\frac{3}{4}\cos\left(\frac{3\pi}{2}\right)$$
$$= -\frac{3}{4} \times 0$$
$$= 0$$

The velocity and acceleration at $t = \pi$ are $\boldsymbol{v = -\frac{3}{2}\ m/s}$ and $\boldsymbol{a = 0\ m/s^2}$.

The velocity and acceleration at $t = 3\pi$ are $\boldsymbol{v = \frac{3}{2}\ m/s}$ and $\boldsymbol{a = 0\ m/s^2}$.

e The equation for acceleration is $a = -\frac{3}{4}\cos\left(\frac{t}{2}\right)$.

The maximum acceleration is $\frac{3}{4}\ \mathrm{m/s^2}$ when $\cos\left(\frac{t}{2}\right) = -1$.

It follows that $\frac{t}{2} = \pi$, that is, $t = 2\pi$.

The particle has the greatest acceleration at $\boldsymbol{t = 2\pi\ s}$.

Question 9

a Use the product rule.

Let $u = e^{\frac{t}{2}}$ and $v = \sin\left(\frac{\pi t}{3}\right)$

Then $u' = \frac{1}{2}e^{\frac{t}{2}}$ and $v' = \frac{\pi}{3}\cos\left(\frac{\pi t}{3}\right)$

$$f'(t) = \frac{d}{dx}(uv) = uv' + vu'$$

$$= e^{\frac{t}{2}} \times \frac{\pi}{3}\cos\left(\frac{\pi t}{3}\right) + \sin\left(\frac{\pi t}{3}\right) \times \frac{1}{2}e^{\frac{t}{2}}$$

$$\boldsymbol{f'(t) = e^{\frac{t}{2}}\left(\frac{\pi}{3}\cos\left(\frac{\pi t}{3}\right) + \frac{1}{2}\sin\left(\frac{\pi t}{3}\right)\right)}$$

b Maximum when $f'(t) = 0$.

Since $e^{\frac{t}{2}} > 0$ for all t, $\frac{\pi}{3}\cos\left(\frac{\pi t}{3}\right) + \frac{1}{2}\sin\left(\frac{\pi t}{3}\right) = 0$

$$\sin\left(\frac{\pi t}{3}\right) = -\frac{2\pi}{3}\cos\left(\frac{\pi t}{3}\right)$$

$$\frac{\sin\left(\frac{\pi t}{3}\right)}{\cos\left(\frac{\pi t}{3}\right)} = -\frac{2\pi}{3}$$

$$\boldsymbol{\tan\left(\frac{\pi t}{3}\right) = -\frac{2\pi}{3}}$$

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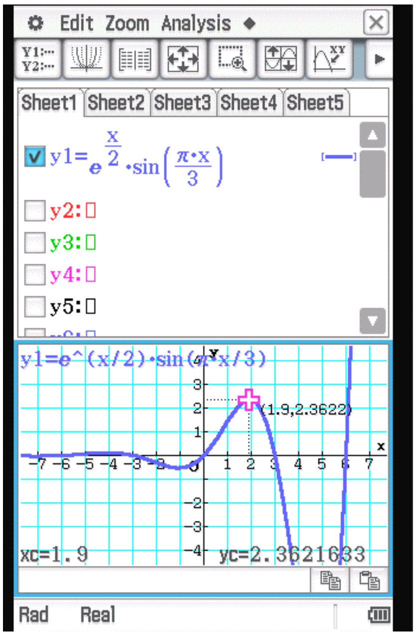
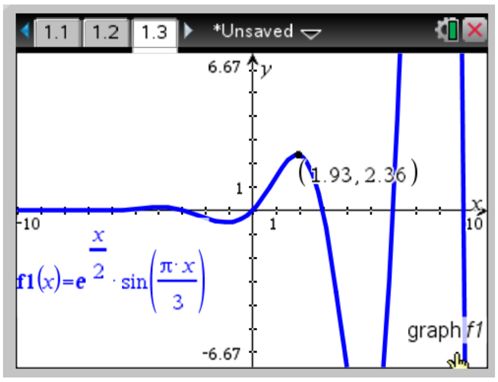
c Use CAS to find the coordinates of the stationary point.

ClassPad **TI-Nspire**

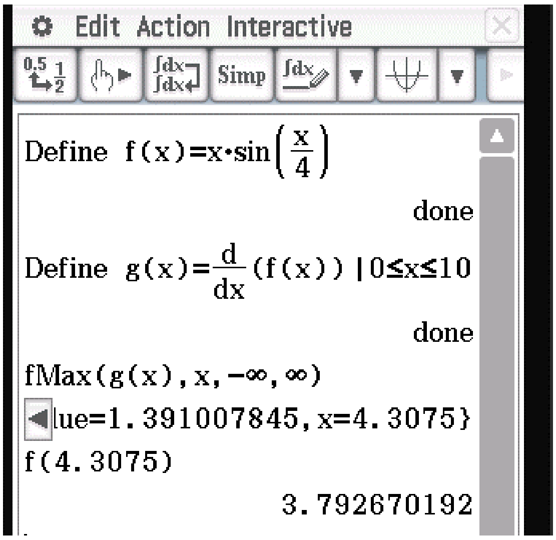




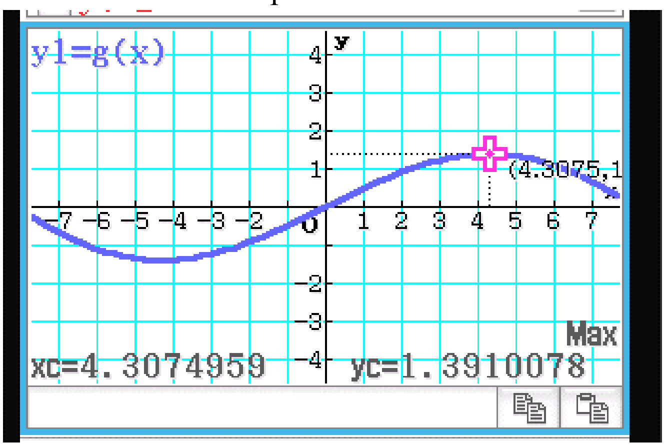
Coordinates of the local maximum (1.9, 2.4).

Question 10

a **ClassPad**

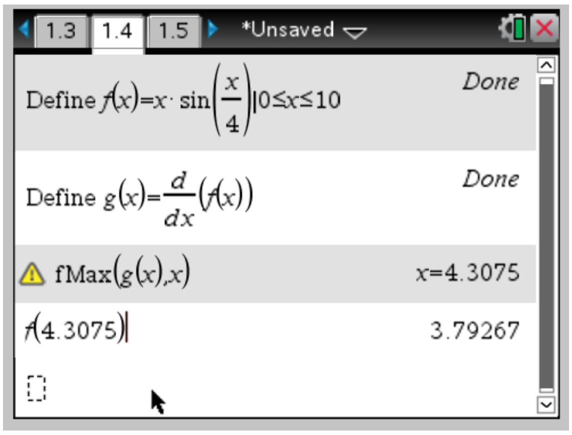


1. Define **f(x)** as shown above.
2. Define **g(x)** as the **derivative** of **f(x)**.
3. Use **Math3** to include the domain **0 ≤ *x* ≤ 10.**
4. Highlight g(x) and tap **Interactive > Calculation >fMin/ fMax > fMax.**
5. The maximum value will be displayed.
6. Copy the answer into **f(x)** to find the y coordinate of the maximum point.

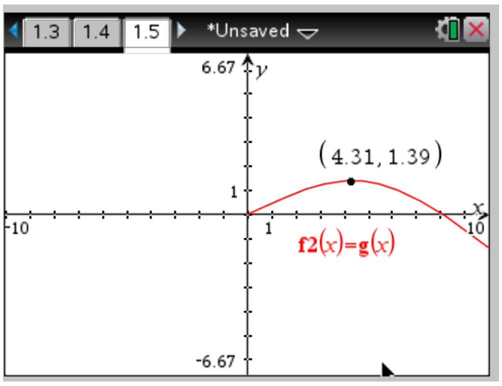


7. To confirm the result, graph g(x) in split screen in **Main**. Include the domain 0 ≤ x ≤ 10.
8. Adjust the window settings to suit.
9. Tap **Analysis > G-Solve > Max**.
10. The coordinates of the maximum point will be displayed.

TI-Nspire



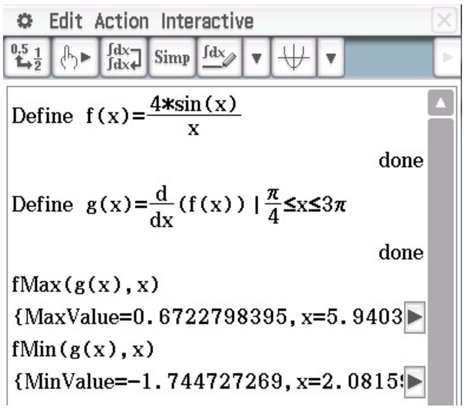
1. Define **f(x)** as shown above.
2. Define **g(x)** as the **derivative** of **f(x)**.
3. Press **catalog** and scroll down to **fMax**.
4. Enter **g(x),x** to find the *x* coordinate of the maximum point.
5. Copy the answer into **f(x)** to find the y coordinate of the maximum point.



6. To confirm the result, graph **g(x)**.
7. Adjust the window settings to suit.
8. Press **menu** >**Analyze Graph** > **Maximum**.
9. When prompted for the **lower bound**, click to the left of the maximum point.
10. When prompted for the **upper bound**, click to the right of the maximum point.

The coordinates of the point experiencing the maximum rate of increase are **(4.31, 3.79)**.

b **ClassPad**



1 Define **f(x)** as shown above.

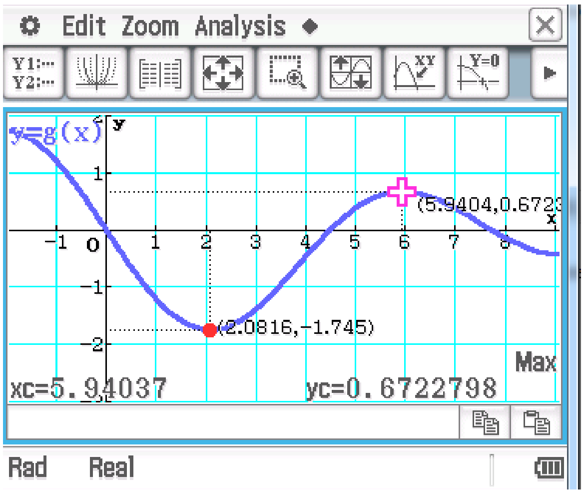
2 Define **g(x)** as the **derivative** of **f(x)**.

3 Use **Math3** to include the domain$|\frac{\pi}{4} \le x \le 3\pi$.

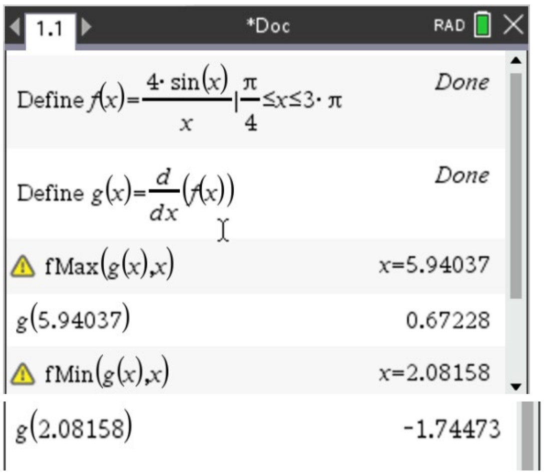
4 Highlight g(x) and tap **Interactive** > **Calculation** >**fMin/ fMax** > **fMax.**

5 The maximum value will be displayed.

6 Repeat by using **fMin** to find the coordinates of the minimum point.



TI-Nspire



1 Define **f(x)** as shown above.

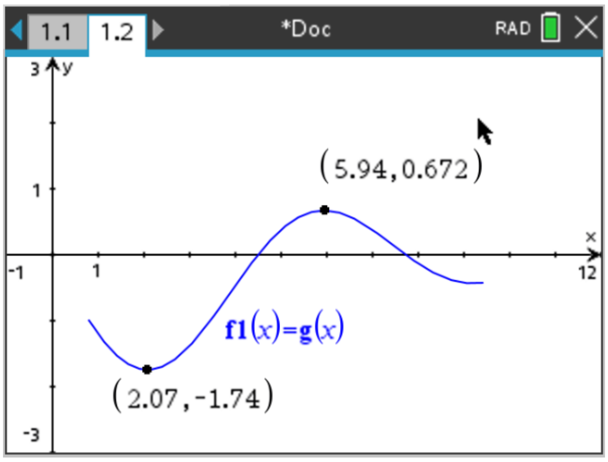
2 Define **g(x)** as the **derivative** of **f(x)**.

3 Press **catalog** and scroll down to **fMax**.

4 Enter **g(x),x** to find the x coordinate of the maximum point.

5 Copy the answer into **g(x)** to find the y coordinate of the maximum point.

6 Repeat by using **fMin** to find the coordinates of the minimum point.



7 To confirm the result, graph g(x) in split screen in **Main**. Include the domain $|\frac{\pi}{4} \leq x \leq 3\pi$.

8 Adjust the window settings to suit.

9 Tap **Analysis** > **G-Solve** > **Min**.

10 The coordinates of the minimum point will be displayed.

11 Tap **Analysis** > **G-Solve** > **fMax** to display the coordinates of the absolute maximum point.

7 To confirm the result, graph **g(x)**.

8 Adjust the window settings to suit.

9 Press **menu** >**Analyze Graph** > **Minimum**.

10 When prompted for the **lower bound**, click to the left of the minimum point.

11 When prompted for the **upper bound**, click to the right of the minimum point.

12 Repeat by using **Analyze Graph** > **Maximum** to find the coordinates of the maximum point.

The coordinates of the point experiencing the maximum positive rate of increase are **(5.94, 0.67)**.

The coordinates of the point experiencing the minimum negative rate of increase are **(2.08, −1.74)**.

Question 11 [SCSA MM2020 Q2] (4 marks)

(✓ = 1 mark)

$$h'(x) = \frac{-e^{-x}\cos(x) - e^{-x}\times(-\sin(x))}{(\cos(x))^2}$$

$$h'(\pi) = \frac{-e^{-\pi}\cos(\pi) - e^{-\pi}\times(-\sin(\pi))}{(\cos(\pi))^2}$$

$$= \frac{-e^{-\pi}\times(-1) - e^{-\pi}\times 0}{(-1)^2}$$

$$= e^{-\pi}$$

demonstrates use of the quotient rule✓

differentiates cos (x) and e^{-x} correctly✓

substitutes $x = \pi$ correctly✓

evaluates correctly✓

Question 12 (3 marks)

(✓ = 1 mark)

a $f(x) = x\sin(x)$

Differentiate the function using the product rule, $\frac{d}{dx}(uv) = uv' + vu'$.

$f'(x) = x\cos(x) + \sin(x)(1)$

The derivative is $\boldsymbol{f'(x) = x\cos(x) + \sin(x)}$.✓

b **Use the quotient rule.**✓

Let $u = x, \quad v = \sin(x)$

Then $u' = 1, v' = \cos(x)$

$$\frac{u'v - uv'}{v^2} = \frac{1\times\sin(x) - x\cos(x)}{(\sin(x))^2}$$

$$= \frac{\sin(x) - x\cos(x)}{\sin^2(x)}$$

$$f'\left(\frac{\pi}{2}\right) = \frac{\sin\left(\frac{\pi}{2}\right) - \left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right)}{\sin^2\left(\frac{\pi}{2}\right)}$$

$$= \frac{1}{1}$$

$$= \mathbf{1}$$ ✓

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Question 13 (2 marks)

(✓ = 1 mark)

The product rule and chain rule are required to obtain the derivative.

Let $u = x^2$ and $v = \sin(2x)$ in the product rule.

Differentiate to obtain u' and v'.

$u' = 2x$

Let $v = \sin(t)$ with $t = 2x$.

Use the chain rule $\frac{dv}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$ to find v'.

$$\begin{aligned}\frac{dv}{dx} &= \frac{dv}{dt} \times \frac{dt}{dx} \\ &= \cos(t)(2) \\ &= 2\cos(2x)\end{aligned}$$

$v' = 2\cos(2x)$

Use the product rule $\frac{dy}{dx} = uv' + vu'$.

$$\begin{aligned}g'(x) &= uv' + vu' \\ &= x^2(2\cos(2x)) + \sin(2x)(2x) \\ &= \mathbf{2x^2\cos(2x) + 2x\sin(2x)} \checkmark\end{aligned}$$

Substitute $x = \frac{\pi}{6}$ into $g'(x)$.

$$\begin{aligned}g'\left(\frac{\pi}{6}\right) &= 2\left(\frac{\pi}{6}\right)^2 \cos\left(2 \times \frac{\pi}{6}\right) + 2\left(\frac{\pi}{6}\right)\sin\left(2 \times \frac{\pi}{6}\right) \\ &= \frac{\pi^2}{18}\cos\left(\frac{\pi}{3}\right) + \frac{\pi}{3}\sin\left(\frac{\pi}{3}\right) \\ &= \frac{\pi^2}{18} \times \frac{1}{2} + \frac{\pi}{3} \times \frac{\sqrt{3}}{2} \\ &= \boldsymbol{\frac{\pi^2}{36} + \frac{\sqrt{3}\pi}{6} = \frac{\pi\sqrt{3}}{6} + \frac{\pi^2}{36}} \checkmark\end{aligned}$$

Question 14 (2 marks)

(✓ = 1 mark)

a Differentiate $y = 4 + 2\cos(3x)$.

$$\frac{dy}{dx} = -6\sin(3x)$$

The tangent at P is parallel to the line $y = 1 - 6x$ whose gradient is −6.

It is known that the gradient of parallel lines is the same. It follows that the gradient at P is −6.

Let P be (x_1, y_1).

$$-6\sin(3x_1) = -6$$
$$\sin(3x_1) = 1$$

It implies that $3x_1 = \frac{\pi}{2}$, that is, $x_1 = \frac{\pi}{6}$.

Obtain y_1 by substituting $x = \frac{\pi}{6}$ into $y = 4 + 2\cos(3x)$.

$$y_1 = 4 + 2\cos\left(3 \times \frac{\pi}{6}\right)$$
$$= 4 + 2\cos\left(\frac{\pi}{2}\right)$$
$$= 4$$

The coordinates of P are $\left(\boldsymbol{\frac{\pi}{6}, 4}\right)$.✓

b Use the point-gradient formula $y - y_1 = m(x - x_1)$ with $m = -6$ and the point $\left(\frac{\pi}{6}, 4\right)$.

$$y - y_1 = m(x - x_1)$$
$$y - 4 = -6\left(x - \frac{\pi}{6}\right)$$
$$y - 4 = -6x + \pi$$
$$y = 4 + \pi - 6x$$

The equation of the tangent line at P is $\boldsymbol{y = 4 + \pi - 6x = -6x + \pi + 4}$.✓

Question 15 [SCSA MM2016 Q4] (8 marks)

(✓ = 1 mark)

a $x = 5\sin 3t$

$$v = \frac{dx}{dt} = 15\cos 3t$$

$$v\left(\frac{\pi}{2}\right) = 15\cos\frac{3\pi}{2} = 0$$

Velocity = 0 micrometres/s

differentiates to determine velocity✓

uses chain rule✓

evaluates velocity at $t = \frac{\pi}{2}$ ✓

b $$\frac{dv}{dt} = \frac{d}{dt}(15\cos 3t)$$

$$= -45\sin 3t$$

$$t = \frac{\pi}{2}, \frac{dv}{dt} = 45$$

Rate of change of velocity = 45 micrometres/s^2

recognises $\frac{dv}{dt}$ as rate of change✓

differentiates velocity✓

evaluates rate at $t = \frac{\pi}{2}$ ✓

c $$\int_0^{\frac{\pi}{2}} \frac{dv}{dt}\,dt = v\left(\frac{\pi}{2}\right) - v(0)$$

$$= 0 - 15$$

$$= -15$$

Integral = −15 micrometres/s

uses Fundamental Theorem of Calculus✓

subtracts velocities at the two limits✓

Question 16 **(3 marks)**

(✓ = 1 mark)

Use the product rule to find the velocity.

$$\frac{dx}{dt} = t\left(-2\sin(2t)\right) + \cos(2t) = -2t\sin(2t) + \cos(2t)$$ ✓

Differentiate a second time to find the acceleration.

$$\frac{d^2x}{dt^2} = -2t\left(2\cos(2t)\right) - 2\sin(2t) - 2\sin(2t)$$ ✓

$$= -4t\cos(2t) - 4\sin(2t)$$

$$t = \frac{\pi}{4}, \frac{d^2x}{dt^2} = -4 \times \frac{\pi}{4}\cos\left(2 \times \frac{\pi}{4}\right) - 4\sin\left(2 \times \frac{\pi}{4}\right)$$

$$= -\pi\cos\frac{\pi}{2} - 4\sin\frac{\pi}{2}$$

$$= 0 - 4$$

$$= -4$$

The acceleration is −4 m/s^2.✓

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Question 17 **[SCSA MM2016 Q21] (6 marks)**

(✓ = 1 mark)

a

$$\tan\theta = \frac{y}{12}$$
$$y = 12\tan\theta$$
$$y = \frac{12\sin\theta}{\cos\theta}$$
$$\frac{dy}{d\theta} = \frac{12\cos\theta\cos\theta + 12\sin\theta\sin\theta}{\cos^2\theta}$$
$$\frac{dy}{d\theta} = \frac{12}{\cos^2\theta}$$

express *y* in terms of tan *θ*✓

differentiates tan *θ*✓

expresses as $\frac{12}{\cos^2\theta}$ equivalent✓

b

$$\frac{dy}{dt} = \frac{dy}{d\theta} \times \frac{d\theta}{dt}$$
$$= \frac{12}{\cos^2\theta} 6\pi = 72\pi(\cos\theta)^{-2}$$

When $y = 5$, $\tan\theta = \frac{5}{12} \Rightarrow \theta \approx 22.62^\circ$ (0.395 radians)

$$\frac{dy}{dt} = 72\pi(\cos\theta)^{-2}$$
$$= 72\pi(\cos(22.62^\circ))^{-2}$$
$$\approx 265.465$$

Velocity = 265.465 kilometres per minute

determines cosθ for $x = 5$✓

uses chain rule with $\frac{d\theta}{dt} = 6\pi$ ✓

determines velocity✓

Question 18 **[SCSA MM2017 Q10] (3 marks)**

(✓ = 1 mark)

$$\frac{d}{dx}\tan(x) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right)$$

$$= \frac{\cos(x)\times\cos(x) - \sin(x)\times(-\sin(x))}{\cos^2(x)}$$

$$= \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)}$$

$$= \frac{1}{\cos^2(x)} \qquad \{\text{since } \cos^2(x)+\sin^2(x)\equiv 1\}$$

writes tangent as a ratio of sine and cosine✓

demonstrates use of the quotient rule✓

states and uses the Pythagorean identity simple results✓

Question 19 (2 marks)

(✓ = 1 mark)

a The curve $y=\sin(x)$ has derivative $y'(x)=\cos(x)$.

At point $P\left(\frac{2\pi}{3},\frac{\sqrt{3}}{2}\right)$, the tangent to the curve has gradient $y'\left(\frac{2\pi}{3}\right)=\cos\left(\frac{2\pi}{3}\right)=-\frac{1}{2}$.

Therefore, $\frac{dy}{dx}$ at $x=\frac{2\pi}{3}$ is $\mathbf{-\frac{1}{2}}$.✓

b Note: 'show that' means that appropriate working must be shown.

The line segment joining P and $C(c,\ 0)$ will have gradient

$$\begin{aligned}m&=\frac{0-\frac{\sqrt{3}}{2}}{c-\frac{2\pi}{3}}\\&=\frac{-\frac{\sqrt{3}}{2}}{\frac{3c-2\pi}{3}}\\&=-\frac{\sqrt{3}}{2}\times\frac{3}{3c-2\pi}\\&=-\frac{3\sqrt{3}}{2(3c-2\pi)}\end{aligned}$$

Also, it is found in part **a** that the gradient at point P is $-\frac{1}{2}$.

So, $-\frac{3\sqrt{3}}{2(3c-2\pi)}=-\frac{1}{2}$.

$$\begin{aligned}3c-2\pi&=3\sqrt{3}\\c&=\frac{3\sqrt{3}+2\pi}{3}\end{aligned}$$

Therefore, the value of c is $\mathbf{\sqrt{3}+\frac{2\pi}{3}}$.✓

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EXERCISE 4.5 Integrals of trigonometric functions

Question 1

Given that $v=\frac{3}{4}\tan(8t-\pi)$. Differentiate v with respect to t to obtain the acceleration.

$$\begin{aligned}a&=v'(t)\\&=\frac{3}{4}\sec^2(8t-\pi)\times 8\\&=\frac{6}{\cos^2(8t-\pi)}\end{aligned}$$

Substitute $t=\frac{5\pi}{32}$ into a.

$$\begin{aligned}a&=\frac{6}{\cos^2\left(8\left(\frac{5\pi}{32}\right)-\pi\right)}\\&=\frac{6}{\cos^2\left(\frac{\pi}{4}\right)}\\&=\frac{6}{\left(\frac{1}{\sqrt{2}}\right)^2}\\&=12\end{aligned}$$

The acceleration is 12 m/s^2.

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Question 2

$$\frac{dy}{dx} = \sqrt{2}\cos(\pi - 3x)$$

$$y = -\frac{\sqrt{2}}{3}\sin(\pi - 3x) + c$$

y will have maximum value when $\sin(\pi - 3x) = -1$.

$$\pi - 3x = -(2n+1)\frac{\pi}{2}, n = 0, 1, 2, ..$$

$$3x = \pi + \frac{(2n+1)}{2}\pi$$

$$x = \frac{(3+2n)\pi}{6}$$

$$n = 0,\ x = \frac{3\pi}{6} = \frac{\pi}{2}$$

Maximum value for $x = \frac{\pi}{2}$.

Question 3

a Consider $f(x) = 6\sin(2x)$. Anti-differentiate.

$$\int 6\sin(2x)\,dx = -6 \times \frac{1}{2}\cos(2x) + c$$

$$= \mathbf{-3\cos(2x) + c}$$

b Consider $f(x) = \frac{1}{2}\cos\left(\frac{x}{2}\right)$. Anti-differentiate.

$$\int \frac{1}{2}\cos\left(\frac{x}{2}\right)dx = \frac{1}{2} \times 2\sin\left(\frac{x}{2}\right) + c$$

$$= \mathbf{\sin\left(\frac{x}{2}\right) + c}$$

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c

Consider $f(x)=3\sin\left(\frac{1}{2}(5x-7)\right)$. Anti-differentiate.

$$\int 3\sin\left(\frac{1}{2}(5x-7)\right)dx = 3\times\left(-\cos\left(\frac{1}{2}(5x-7)\right)\right)\times\frac{2}{5}+c$$

$$= \mathbf{-\frac{6}{5}\cos\left(\frac{1}{2}(5x-7)\right)+c}$$

d

$$\int_0^{\frac{\pi}{12}} 2\cos(2x)\,dx = \left[\sin(2x)\right]_0^{\frac{\pi}{12}}$$

$$= \sin\left(2\times\frac{\pi}{12}\right)-\sin(0)$$

$$= \sin\left(\frac{\pi}{6}\right)-0$$

$$= \mathbf{\frac{1}{2}}$$

e

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} 4\sin(2x)\,dx = \left[-2\cos(2x)\right]_{\frac{\pi}{12}}^{\frac{\pi}{6}}$$

$$= -2\cos\left(2\times\frac{\pi}{6}\right)-\left(-2\cos\left(2\times\frac{\pi}{12}\right)\right)$$

$$= -2\cos\left(\frac{\pi}{3}\right)+2\cos\left(\frac{\pi}{6}\right)$$

$$= -2\times\frac{1}{2}+2\times\frac{\sqrt{3}}{2}$$

$$= \sqrt{3}-1$$

f

$$\int_0^{\frac{\pi}{12}} \sin(2x)-\cos(3x)\,dx = \left[-\frac{1}{2}\cos(2x)-\frac{1}{3}\sin(3x)\right]_0^{\frac{\pi}{12}}$$

$$= -\frac{1}{2}\cos\left(2\times\frac{\pi}{12}\right)-\frac{1}{3}\sin\left(3\times\frac{\pi}{12}\right)-\left(-\frac{1}{2}\cos(2\times 0)-\frac{1}{3}\sin(3\times 0)\right)$$

$$= -\frac{1}{2}\cos\left(\frac{\pi}{6}\right)-\frac{1}{3}\sin\left(\frac{\pi}{4}\right)-\left(-\frac{1}{2}-0\right)$$

$$= -\frac{1}{2}\times\frac{\sqrt{3}}{2}-\frac{1}{3}\times\frac{1}{\sqrt{2}}+\frac{1}{2}$$

$$= \mathbf{-\frac{\sqrt{3}}{4}-\frac{1}{3\sqrt{2}}+\frac{1}{2}}$$

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Question 4

$$\int \frac{d}{dx}\left(3x\sin(2x)\right)dx = \int\left(3\sin(2x)+6x\cos(2x)\right)dx$$

$$3x\sin(2x) = -\frac{3}{2}\cos(2x)+\int 6x\cos(2x)\,dx$$

$$\int 6x\cos(2x)\,dx = \frac{3}{2}\cos(2x)+3x\sin(2x)$$

$$\boldsymbol{\int 12x\cos(2x)\,dx = 3\cos(2x)+6x\sin(2x)+c}$$

Question 5

$$f(x) = \int\left(\sin(3x)+\cos(x)\right)dx + c$$

$$= -\frac{1}{3}\cos(3x)+\sin(x)+c$$

$$f\left(\frac{\pi}{2}\right) = 3$$

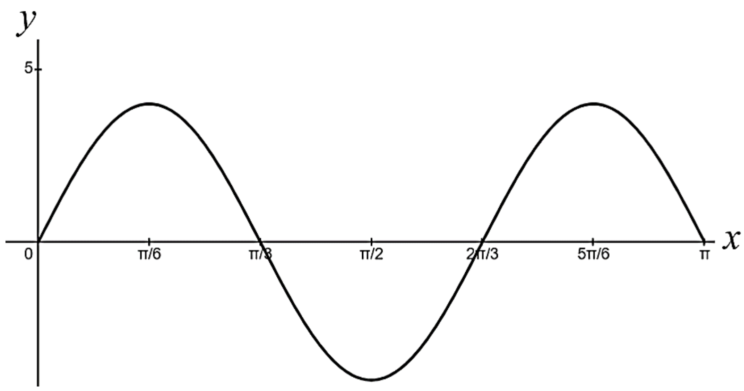
$$-\frac{1}{3}\cos\left(3\times\frac{\pi}{2}\right)+\sin\left(\frac{\pi}{2}\right)+c = 3$$

$$0+1+c=3$$

$$c=2$$

$$\boldsymbol{f(x) = -\frac{1}{3}\cos(3x)+\sin(x)+2 = \sin(x)-\frac{1}{3}\cos(3x)+2}$$

Question 6



Area is $3\int_0^{\frac{\pi}{3}} 4\sin(3x)\,dx$

$$\begin{aligned}3\int_0^{\frac{\pi}{3}} 4\sin(3x)\,dx &= 3\times 4\left[-\frac{1}{3}\cos(3x)\right]_0^{\frac{\pi}{3}}\\ &= 4\left(-\cos(\pi)+\cos(0)\right)\\ &= 4(1+1)\\ &= 8\end{aligned}$$

Area = 8 units²

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Question 7

a

$$a = \frac{dv}{dt} = \frac{\pi}{40} \times \frac{7\pi}{8}\cos\left(\frac{\pi t}{40}\right)$$
$$= \boldsymbol{\frac{7\pi^2}{320}\cos\left(\frac{\pi t}{40}\right)}$$

b

$$\int_0^{\frac{20}{3}} v'(t)\,dt = \left[v(t)\right]_0^{\frac{20}{3}}$$
$$= \left[\frac{7\pi}{8}\sin\left(\frac{\pi t}{40}\right)\right]_0^{\frac{20}{3}}$$
$$= \frac{7\pi}{8}\left(\sin\left(\frac{\pi}{6}\right) - \sin(0)\right)$$
$$= \frac{7\pi}{8}\left(\frac{1}{2}\right)$$
$$= \boldsymbol{\frac{7\pi}{16}}$$

c

$$x(t) = \int \frac{7\pi}{8}\sin\left(\frac{\pi t}{40}\right)dt$$
$$= -\frac{7\pi}{8} \times \frac{40}{\pi}\cos\left(\frac{\pi t}{40}\right) + c$$
$$= -35\cos\left(\frac{\pi t}{40}\right) + c$$

$t = 0,\ x(0) = 0$

$0 = -35\cos(0) + c$

$c = 35$

$$\boldsymbol{x(t) = 35 - 35\cos\left(\frac{\pi t}{40}\right)}$$

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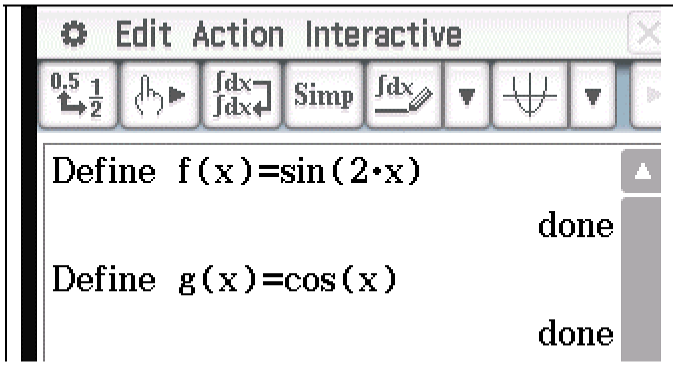
Question 8

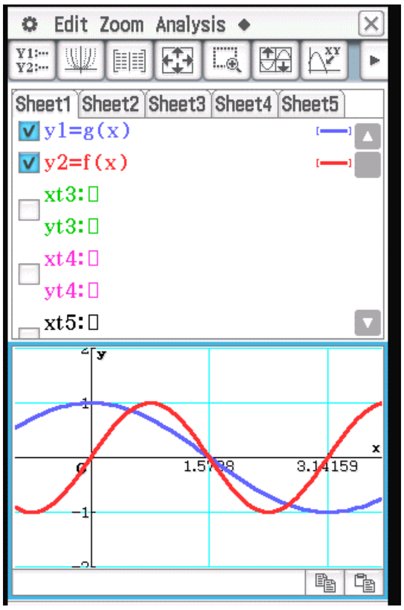
ClassPad

1 In Main, define f(x) = sin (2*x*) and g(x) = cos (*x*).



2 Open the graph screen, highlight and drag down *f* (*x*) and *g*(*x*).

3 Adjust the window settings to suit $0 \le x \le \pi$.



4 Tap Analysis > G-Solve > Integral > ∫dx Intersection.

5 When the first point of intersection is displayed, press EXE, then tap the right arrow.

6 When the second point of intersection is displayed, press EXE.

7 The shaded area between the curves and the value will be displayed.

TI-Nspire

1 Graph f1(x) and f2(x) as shown above.

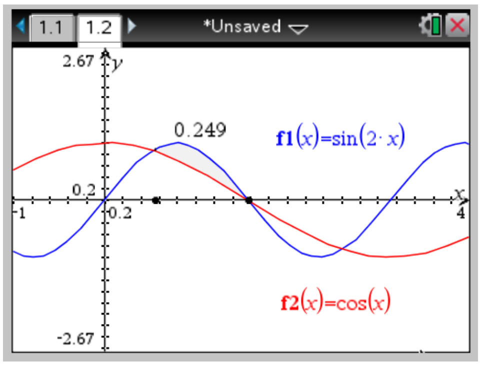
2 Adjust the window settings to suit.

3 Press menu > Analyze Graph > Bounded Area.

4 When prompted for the lower bound, click on the first point of intersection.

5 When prompted for the upper bound, click on the second point of intersection.

6 The area will be displayed.



7 To find the exact area, you need to solve sin(2x) = cox(x) over the domain $0 \le x \le \dfrac{\pi}{2}$.

8 Press menu > Calculus > Integral.

9 Enter the two smaller solutions as the lower and upper limits, and enter the expression as shown above.

10 The exact area will be displayed.

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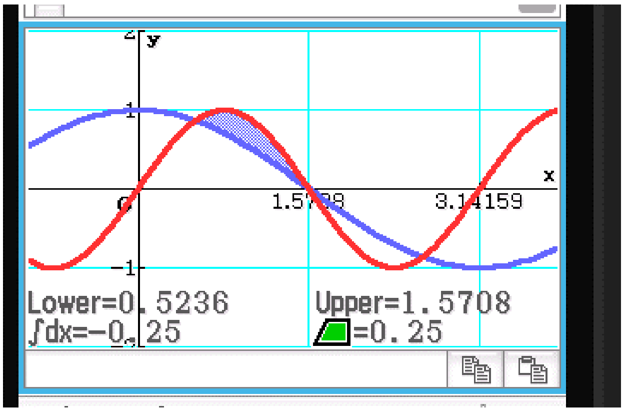
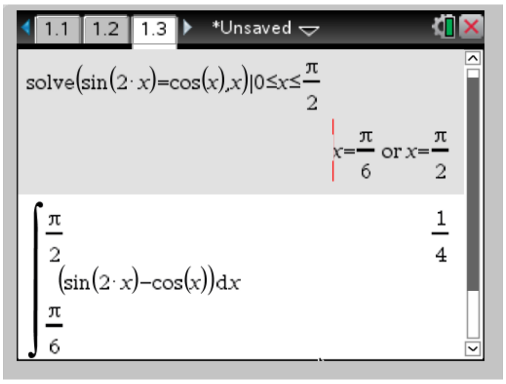




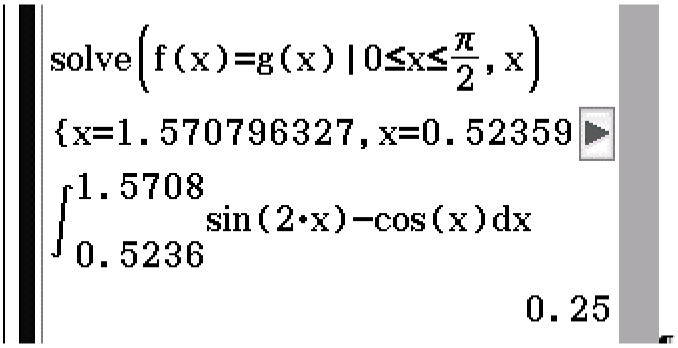
8 To find the exact area, solve the equation sin (2x)=cos (x) or f(x)=g(x), over the domain $0 \le x \le \dfrac{\pi}{2}$.

9 Highlight sin (2x)-cos (x) or f(x)-g(x) and tap Interactive > Calculation > ∫.

10 In the dialogue box, tap Definite.

11 Enter the two solutions as the lower and upper limits, and enter the expression as shown above.

12 The exact area will be displayed.



Area = **0.25 units^2**

Question 9 (4 marks)

(✓ = 1 mark)

a Use $\int \cos(ax-b)\,dx = \frac{1}{a}\sin(ax-b)+c$ with $a=2$ and $b=-1$

$$\int \cos(2x+1)\,dx = \frac{1}{2}\sin(2x+1)+c \checkmark$$

b
$$f(x) = \int (2\cos(x) - \sin(2x))\,dx$$
$$= 2\sin(x) + \frac{1}{2}\cos(2x) + c \checkmark$$

$$f\left(\frac{\pi}{2}\right) = \frac{1}{2}$$

$$2\sin\left(\frac{\pi}{2}\right) + \frac{1}{2}\cos(\pi) + c = \frac{1}{2}$$

$$2 - \frac{1}{2} + c = \frac{1}{2}$$

$$c = -1 \checkmark$$

$$f(x) = 2\sin(x) + \frac{1}{2}\cos(2x) - 1 \checkmark$$

Question 10 (3 marks)

(✓ = 1 mark)

$$\int f'(x)\,dx = \int (\cos(3x) - 3x\sin(3x))\,dx = x\cos(3x) \checkmark$$

$$\int (\cos(3x) - 3x\sin(3x))\,dx = x\cos(3x)$$

$$\frac{1}{3}\sin(3x) - \int 3x\sin(3x)\,dx = x\cos(3x)$$

$$\int 3x\sin(3x)\,\mathrm{dx} = \frac{1}{3}\sin(3x) - x\cos(3x) \checkmark$$

$$\int x\sin(3x)dx = \frac{1}{9}\sin(3x) - \frac{1}{3}x\cos(3x) + c = \frac{\sin(3x)}{9} - \frac{x\cos(3x)}{3} + c \checkmark$$

Question 11 (2 marks)

(✓ = 1 mark)

$$v(t) = \int a(t)\,dt = \int \left(2\sin(t) - 18\cos(3t)\right)dt$$

$$v(t) = -2\cos(t) - 6\sin(3t) + c$$

$$v(\pi) = 0$$

$$-2\cos(\pi) - 6\sin(3\pi) + c = 0$$

$$2 + c = 0 \Rightarrow c = -2$$

$$\boldsymbol{v(t) = -2\cos(t) - 6\sin(3t) - 2} \checkmark$$

$$s(t) = \int v(t)\,dt = \int \left(-2\cos(t) - 6\sin(3t) - 2\right)dt$$

$$s(t) = -2\sin(t) + 2\cos(3t) - 2t + c$$

$$s\left(\frac{\pi}{2}\right) = -\pi$$

$$-2\sin\left(\frac{\pi}{2}\right) + 2\cos\left(\frac{3\pi}{2}\right) - 2\left(\frac{\pi}{2}\right) + c = -\pi$$

$$-2 - \pi + c = -\pi$$

$$c = 2$$

$$\boldsymbol{s(t) = -2\sin(t) + 2\cos(3t) - 2t + 2} \checkmark$$

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Question 12 (2 marks)

(✓ = 1 mark)

$v(t)=\int a(t)\,dt=\int \cos(t)\,dt=\sin(t)+c_1$ ✓

$s(t)=\int v(t)\,dt=\int(\sin(t)+c_1)\,dt$

$s(t)=-\cos(t)+c_1t+c_2$

$s\left(\frac{\pi}{2}\right)=\frac{\pi}{2}+2\Rightarrow-\cos\left(\frac{\pi}{2}\right)+c_1\times\frac{\pi}{2}+c_2=\frac{\pi}{2}+2$ or $\pi c_1+2c_2=\pi+4$

$s(\pi)=\pi+3\Rightarrow-\cos(\pi)+c_1\times\pi+c_2=\pi+3$ or $\pi c_1+c_2=\pi+2$

Use a calculator of algebra to solve for c_1 and c_2.

$c_1=1, c_2=2$

Hence $\boldsymbol{s(t)=-\cos(t)+t+2}$ ✓

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Question 13 **[SCSA MM2017 Q8] (5 marks)**

(✓ = 1 mark)

a

$$\frac{d\left(2x\sin(3x)\right)}{dx}$$
$$= 2\times\sin(3x) + 2x\times 3\cos(3x)$$
$$= 2\sin(3x) + 6x\cos(3x)$$

uses product rule✓

obtains correct answer✓

b

$$\frac{d\left(2x\sin(3x)\right)}{dx} = 2\sin(3x) + 6x\cos(3x)$$
$$\int \frac{d\left(2x\sin(3x)\right)}{dx}\,dx = \int\left(2\sin(3x) + 6x\cos(3x)\right)dx$$
$$2x\sin(3x) + c_1 = \int 2\sin(3x)\,dx + 6\int x\cos(3x)\,dx$$
$$\frac{2x\sin(3x) + c_1}{6} = -\frac{2\cos(3x)}{18} + c_2 + \int x\cos(3x)\,dx$$
$$\int x\cos(3x)\,dx = \frac{2x\sin(3x)}{6} + \frac{2\cos(3x)}{18} + c$$
$$\therefore \int x\cos(3x)\,dx = \frac{3x\sin(3x) + \cos(3x)}{9} + c$$

integrates both sides✓

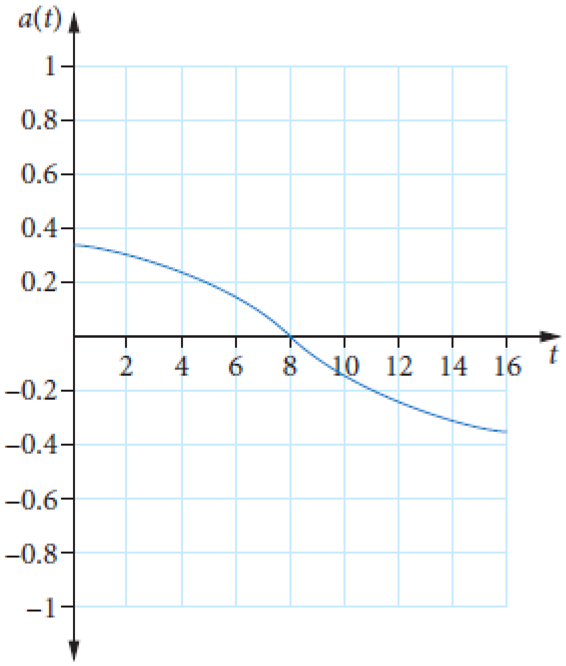
uses Fundamental Theorem of Calculus to simplify LHS✓

evaluates integrals and rearranges to show results✓

Question 14 **[SCSA MM2019 Q9] (8 marks)**

(✓ = 1 mark)

a $a(t)=\frac{9\pi^2}{256}\cos\left(\frac{\pi t}{16}\right)\ \text{m}^2/\text{s}$



differentiates the velocity function to obtain the acceleration✓

provides a sketch clearly showing the peak acceleration values and correctly locating the zero acceleration point✓

b Since the acceleration is positive on the interval $0 < t < 8$, the velocity is increasing on the interval $0 < t < 8$.

Since the acceleration is negative on the interval $8 < t < 16$ the velocity is decreasing on the interval $8 < t < 16$.

references acceleration graph or function✓

recognises that the upward velocity is increasing on the interval 0 < t < 8✓

recognises that the upward velocity is decreasing on the interval 8 < t < 16✓

c The displacement is the integral of the velocity

$$x(t) = -9\cos\left(\frac{\pi t}{16}\right) + c \text{ m}$$

Since $x(0) = 0$ it follows that

$$0 = -9\cos(0) + c$$
$$0 = -9 + c$$
$$c = 9$$

Hence

$$x(t) = 9 - 9\cos\left(\frac{\pi t}{16}\right)$$

Evaluating $x(16)$

$$x(16) = 9 - 9\cos(\pi)$$
$$= 18 \text{ m}$$

integrates the velocity function to obtain the displacement including unknown integration constant✓

determines integration constant✓

evaluates $x(16)$✓

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Question 15 **[SCSA MM2020 Q17ab] (8 marks)**

(✓ = 1 mark)

a

$$A(p,q)=\int_0^q\left(10\sin\left(\frac{x}{15}\right)+p\right)dx$$

$$=pq-150\cos\left(\frac{q}{15}\right)+150$$

$$p+q=500$$

$$\therefore p=500-q$$

$$A(q)=q(500-q)-150\cos\left(\frac{q}{15}\right)+150$$

$$=500q-150\cos\left(\frac{q}{15}\right)-q^2+150$$

states correct integral for area✓

evaluates integral to determine equation for area in terms of *p* and *q*✓

states that *p* + *q* = 500✓

substitutes for *p* to obtain the required result✓

b

$$A'(q)=500+10\sin\left(\frac{q}{15}\right)-2q$$

$$0=500+10\sin\left(\frac{q}{15}\right)-2q$$

$$q\approx 246.65$$

$$A''(q)=\frac{2}{3}\cos\left(\frac{q}{15}\right)-2$$

$$A''(246.65)=-ve\{-2.02\}$$

$\therefore$ maximum

$$A(246.65)=62\ 750$$

$\therefore$ maximum area = 62 750 m^2

differentiates the area equation✓

sets the derivative to 0 and solves it to obtain *q*✓

obtains the second derivative (or draws a sign diagram for the derivative) to conclude that the point is a global maximum✓

states the maximum area✓

Question 16 **[SCSA MM2021 Q9abc] (5 marks)**

(✓ = 1 mark)

a $h(W) = 0$

$W = 8.64$ m (or 864 cm)

sets $h(W) = 0$✓

solves for W✓

b $$h'(x) = 4\cos\left(x - \frac{3\pi}{2}\right) - 2x + 3\pi$$

differentiates $h(x)$✓

c Setting $h'(x) = 0$ gives $x = 5.74$ m

Hence the maximum height $h(5.74) = 20.57$m

sets $h'(x) = 0$ and solves it to obtain the value of x for maximum height✓

states the maximum height✓

Question 17 **(2 marks)**

(✓ = 1 mark)

Calculate the area of the 3 rectangles and add them.

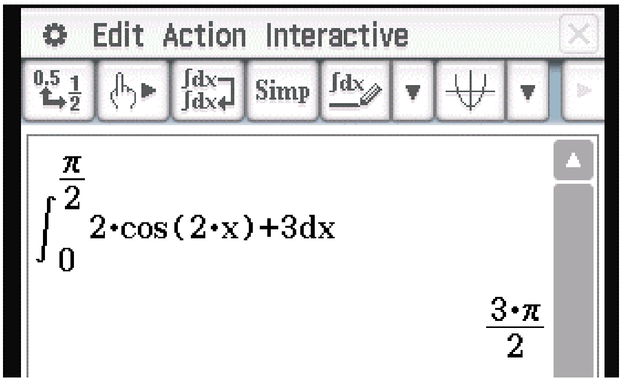
x	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$f(x)$	4	2	1

From the diagram in the question and the table above, the areas of the 3 rectangles added together are:

$$\left(4\times\frac{\pi}{6}\right)+\left(2\times\frac{\pi}{6}\right)+\left(1\times\frac{\pi}{6}\right)=\frac{7\pi}{6}$$ ✓

Using CAS to find the actual area by integration.

ClassPad



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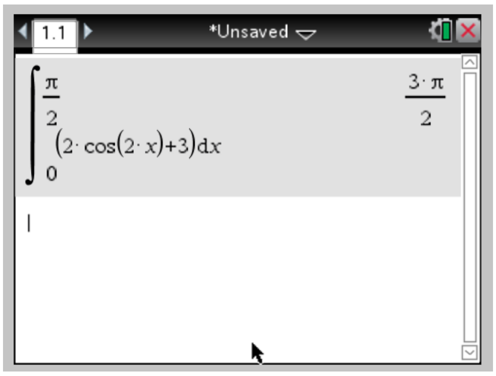


Actual area = $\frac{3\pi}{2}$

The ratio of areas is $\frac{\frac{7\pi}{6}}{\frac{3\pi}{2}}=\frac{7\pi}{\cancel{6}_3}\times\frac{\cancel{2}}{3\pi}=\frac{7}{9}$ ✓

Cumulative examination: Calculator-free

Question 1 (6 marks)

(✓ = 1 mark)

a $y=(x+1)(x^2-3x+3)$

Find $\frac{dy}{dx}$ using the product rule.

$\boldsymbol{u=x+1,\quad v=x^2-3x+3}$ ✓

$\frac{du}{dx}=1,\quad \frac{dv}{dx}=2x-3$

$\frac{dy}{dx}=(1)(x^2-3x+3)+(x+1)(2x-3)$

At $x=3$,

$\boldsymbol{\frac{dy}{dx}=(3^2-3(3)+3)+(3+1)(2(3)-3)}$ ✓

$=3+12$

$\boldsymbol{=15}$ ✓

b At $x=3$,

$\frac{dy}{dx}=x^2-3x+3+(x+1)(2x-3)$

$=x^2-3x+3+2x^2-x-3$

$=3x^2-4x$

$\boldsymbol{=x(3x-4)}$ ✓

For $0<x<\frac{4}{3}$, $\frac{dy}{dx}<0$, and for $x>\frac{4}{3}$, $\frac{dy}{dx}>0$. ✓

Thus there is a minimum turning point at $x=\frac{4}{3}$. ✓

Question 2 **[SCSA MM2018 Q3a-ci] (7 marks)**

(✓ = 1 mark)

a Differentiate $(2x^3 + 1)^5$

Let $u = 2x^3 + 1$

Then $y = (2x^3 + 1)^5$

$= u^5$

Write the chain rule and find the two derivatives.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{du} = 5u^4$$

$$\frac{du}{dx} = 6x^2 \checkmark$$

Substitute the derivatives.

$$\frac{dy}{dx} = 5u^4\ 6x^2$$

Substitute for u.

$$\frac{dy}{dx} = \mathbf{5(2x^3 + 1)^4\ (6x^2)}$$

$$\mathbf{= 30x^2(2x^3 + 1)^4} \checkmark$$

b $g'(x) = e^{2x}\sin(3x)$

$g''(x) = 3e^{2x}\cos(3x) + 2e^{2x}\sin(3x)$ ✓

$$g''\left(\frac{\pi}{2}\right) = 3e^{2\left(\frac{\pi}{2}\right)}\cos\left(3\times\frac{\pi}{2}\right) + 2e^{2\left(\frac{\pi}{2}\right)}\sin\left(3\times\frac{\pi}{2}\right)$$

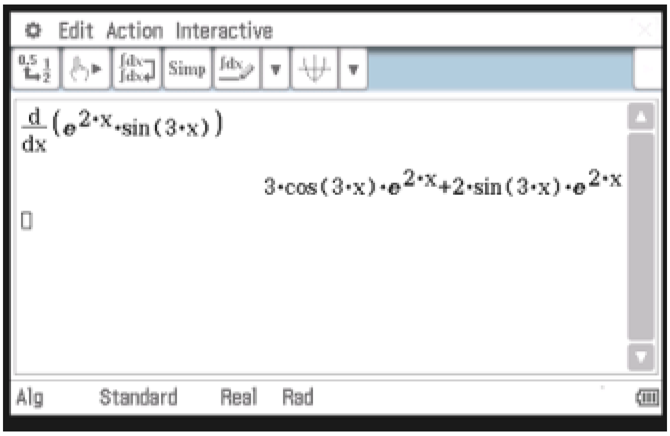
$$= 3e^{\pi}\cos\left(\frac{3\pi}{2}\right) + 2e^{\pi}\sin\left(\frac{3\pi}{2}\right)$$

$$= e^{\pi}\left[3\times 0 + 2\times -1\right]$$

$$= -2e^{\pi}$$

✓✓

ClassPad



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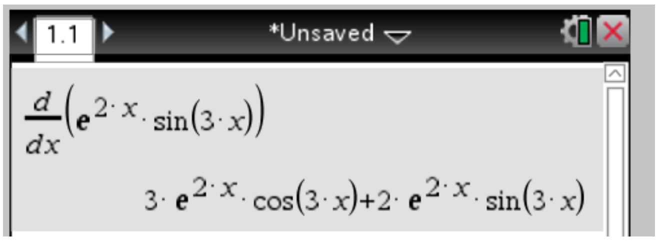


c $\int 3\cos(2x)\,dx$

$= 3\int \cos(2x)\,dx$

$= 3 \times \frac{1}{2}\sin(2x) + c$

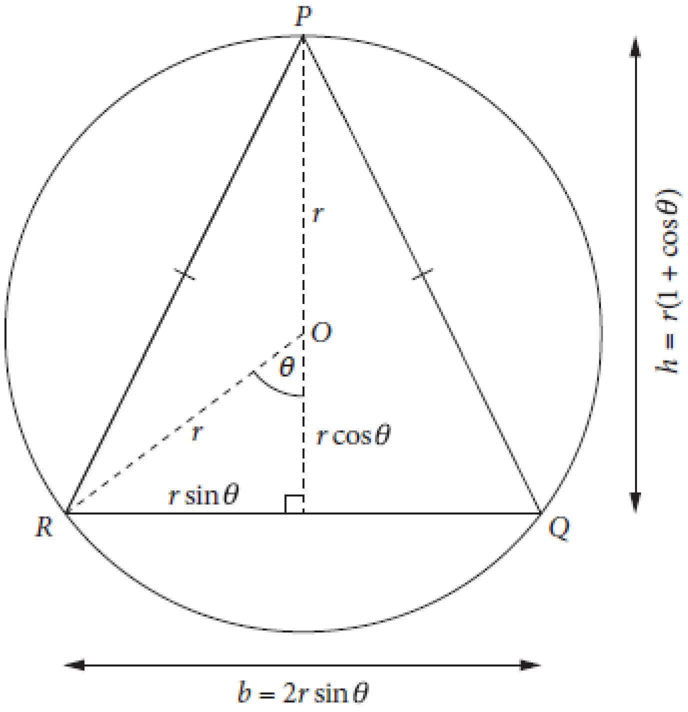
$= \frac{3}{2}\sin(2x) + c$

✓✓

Question 3 **[SCSA MM2016 Q8] (7 marks)**

(✓ = 1 mark)

a



Area is $\frac{1}{2} \times 2r\sin\theta \times r(1+\cos\theta) = r^2\sin\theta(1+\cos\theta)$

determines an expression of height in terms of *r*, *θ*✓

determines an expression for base in terms of *r*, *θ*✓

b $A = r^2 \sin\theta(1+\cos\theta)$

$$\frac{dA}{d\theta} = r^2\left[\sin\theta(-\sin\theta)+(1+\cos\theta)\cos\theta\right]$$

$$\frac{dA}{d\theta} = r^2\left[\cos\theta+\cos^2\theta-\sin^2\theta\right]$$

$$\frac{dA}{d\theta} = r^2\left[\cos\theta+\cos^2\theta-\left(1-\cos^2\theta\right)\right]$$

$$\frac{dA}{d\theta} = r^2\left[2\cos^2\theta+\cos\theta-1\right] = r^2(2\cos\theta-1)(\cos\theta+1)$$

$$\frac{dA}{d\theta} = 0 \quad \cos\theta = \frac{1}{2}, \theta = \frac{\pi}{3} \quad ,\cos\theta \neq -1 \quad ,0<\theta<\pi$$

$$A = r^2\sin\theta(1+\cos\theta) = r^2\frac{\sqrt{3}}{2}\left(1+\frac{1}{2}\right) = \frac{3\sqrt{3}}{4}r^2$$

differentiates area with respect to θ using calculus✓

equated derivative to zero to solve for optimal value✓

rearranges derivative to allow solving for θ exactly✓

solves for $0 < \theta < \pi$ allowing for one solution only✓

states exact area for this optimal value✓

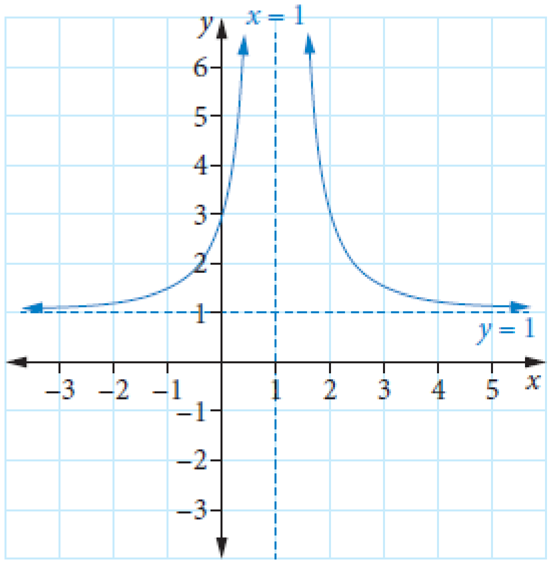
Cumulative examination: Calculator-assumed

Question 1 **(5 marks)**

(✓ = 1 mark)

a **i** $f(-1)=\dfrac{2}{(-1-1)^2}+1=\dfrac{3}{2}$ ✓

ii



The function is undefined for $x = 1$, there is a vertical asymptote at $x = 1$.

As the values of x increase in the positive direction, the value of $f(x)$ approaches 1.

As the values of x increase in the negative direction, the value of $f(x)$ approaches 1.

Hence, $y = 1$ is a horizontal asymptote.

$f(0) = 3$, thus the y-intercept is at (0, 3).

correct asymptotes and *y*-intercept✓

branches of the graph drawn accurately✓

b $$\int_{-1}^{0}\left(\frac{2}{(x-1)^2}+1\right)dx=\int_{-1}^{0}\left(2(x-1)^{-2}+1\right)dx \checkmark$$

$$=\left[-\frac{2}{x-1}+x\right]_{-1}^{0}$$

$$=-\frac{2}{-1}-\left(-\frac{2}{-1-1}-1\right)$$

$$=2$$

The area is 2 units^2.✓

Question 2 **[SCSA MM2017 Q16] (8 marks)**

(✓ = 1 mark)

a

$$P(t) = 2300e^{0.065t}$$

$$P(3) = 2300e^{0.065(3)}$$

$$= 2795.2$$

$$\approx 2800$$

determines equation for *P*✓

determines population correct to nearest 10✓

b

$$\frac{dP}{dt} = 0.065 \times 2300e^{0.065t}$$

$$\frac{dP}{dt} = 149.5e^{0.065t}$$

$$\left.\frac{dP}{dt}\right|_{t=2.5} = 175.879$$

≈ 176 animals/year

determines derivative✓

determines rate at 2.5 years✓

c

$$P(6) = 2300e^{0.065(6)}$$

$$\approx 3397$$

Population from 2017:

$$P(t) = 3397e^{-0.0825t}$$

$$1000 = 3397e^{-0.0825t}$$

$$t = 14.8$$

October 2031

determines population at the beginning of 2017✓

states new population equation✓

solves for *t*✓

determines correct month and year✓

Question 3 [SCSA MM2018 Q11] (8 marks)

(✓ = 1 mark)

a

$$\int 2\sin\left(\frac{t}{3}+\frac{\pi}{6}\right)dt$$

$$=-6\cos\left(\frac{t}{3}+\frac{\pi}{6}\right)+c$$

Solve for $t = 0$: $-6\cos\left(\frac{\pi}{6}\right)+c=0$

$c = 3\sqrt{3}$ or 5.196152423

$\therefore x(t)=-6\cos\left(\frac{t}{3}+\frac{\pi}{6}\right)+3\sqrt{3}$ or $x(t)=-6\cos\left(\frac{t}{3}+\frac{\pi}{6}\right)+5.196$

integrates $v(t)$ to determine cosine expression✓

recognises $x(t)$ involves a constant term and equates $x(0)$ to 0✓

solves for c and states $x(t)$✓

b

$$x(16)=-6\cos\left(\frac{16}{3}+\frac{\pi}{6}\right)+3\sqrt{3}$$

$$=-0.266975$$

The drone is 0.27 m (27 cm) due south of the pilot.

evaluates displacement at $t = 16$✓

interprets solution✓

c

$$a(t)=\frac{2}{3}\cos\left(\frac{t}{3}+\frac{\pi}{6}\right)$$

$$-0.5=\frac{2}{3}\cos\left(\frac{t}{3}+\frac{\pi}{6}\right)$$

$t = 5.6858$ or 10.0222

heading south at $t = 10.0222$

distance travelled $= \int_0^{10.0222}\left|2\sin\left(\frac{t}{3}+\frac{\pi}{6}\right)\right|dt = 12.696$

The drone has travelled 12.696 metres.

equates derivative to −0.5 m/s^2✓

recognises 10.0222 is when the drone is heading south✓

determines distance travelled✓

Chapter 5 – Discrete random variables

EXERCISE 5.1 Review of probability

Question 1

On each selection, the bulb selected can either be a tulip bulb (T) or a daffodil bulb (D).

The first selection is made from 18 tulip bulbs and 12 daffodil bulbs.

On the first selection:

$n(T) = 18$, $n(D) = 12$

$$P(T) = \frac{18}{30},\ P(D) = \frac{12}{30}$$

In the second selection, the number of tulip bulbs and daffodil bulbs is determined by the type of bulb selected first.

On the second selection:

If the first bulb selected is a tulip:

$n(T) = 17$, $n(D) = 12$

$$P(T) = \frac{17}{29},\ P(D) = \frac{12}{29}$$

If the first bulb selected is a daffodil:

$n(T) = 18$, $n(D) = 11$

$$P(T) = \frac{18}{29},\ P(D) = \frac{11}{29}$$

In the third selection, the number of tulip bulbs and daffodil bulbs is determined by the type of bulb selected first and second.

On the third selection:

If the first and second bulb selected is a tulip:

$n(T) = 16$, $n(D) = 12$

$$P(T) = \frac{16}{28},\ P(D) = \frac{12}{28}$$

If the first bulb selected is a tulip and the second bulb selected is a daffodil:

$n(T) = 17$, $n(D) = 11$

$$P(T) = \frac{17}{28},\ P(D) = \frac{11}{28}$$

If the first bulb selected is a daffodil and the second bulb selected is a tulip:

$n(T) = 17, n(D) = 11$

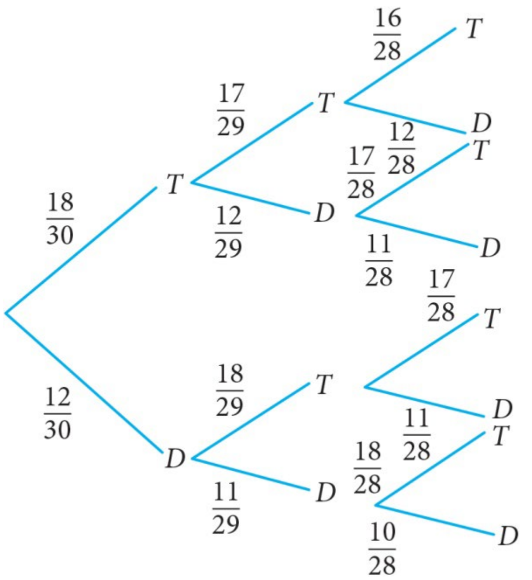
$$P(T)=\frac{17}{28},\ P(D)=\frac{11}{28}$$

If the first and second bulb selected is a daffodil:

$n(T) = 18, n(D) = 10$

$$P(T)=\frac{18}{28},\ P(D)=\frac{10}{28}$$

Next, we represent these probabilities with a tree diagram.



Now, we identify the branches where three bulbs are of the same type.

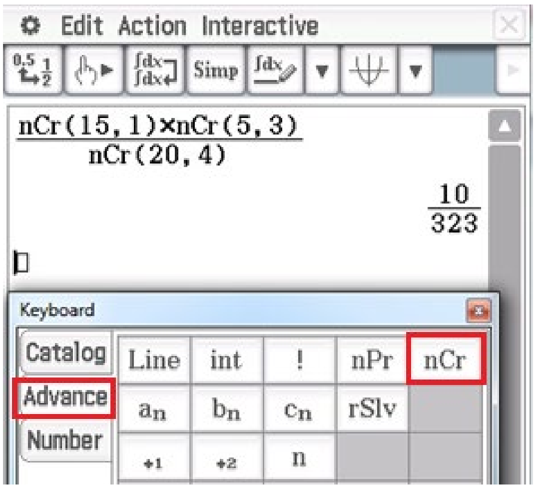
P(3 bulbs of the same type) = P(*TTT*) + P(*DDD*)

Finally, we multiply the probabilities along the branches and add the products.

$$\text{P(3 bulbs of the same type)} = \frac{18}{30}\times\frac{17}{29}\times\frac{16}{28}+\frac{12}{30}\times\frac{11}{29}\times\frac{10}{28}=\mathbf{\frac{37}{145}}$$

Question 2

ClassPad

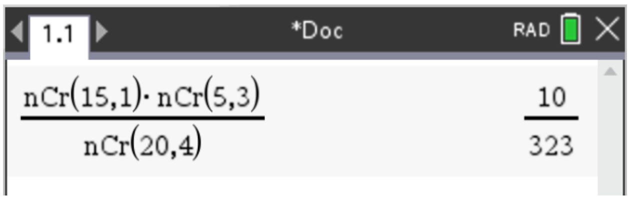


1 Open the soft Keyboard and tap the downward arrow.

2 Tap **Advance** and find the **nCr** symbol.

3 Enter the combinations and values as shown above.

TI-Nspire



1 Press **menu** > **Probability** > **Combinations**.

2 Enter the combinations and values as shown above.

The probability of choosing three green marbles is $\mathbf{\frac{10}{323}}$.

Question 3

a

2nd roll		1st roll: 2	4	6	8
	8	(2, 8)	(4, 8)	(6, 8)	(8, 8)
	6	(2, 6)	(4, 6)	(6, 6)	(8, 6)
	4	(2, 4)	(4, 4)	(6, 4)	(8, 4)
	2	(2, 2)	(4, 2)	(6, 2)	(8, 2)

Draw a 4 × 4 grid and let the columns represent the first roll of the die and let the rows represent the second roll of the die.

All cells are used, representing the 16 possible combinations. For example, (4, 6) means the first roll was a 4 and the second roll was a 6.

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b i

2nd roll				
8				(8, 8)
6			(6, 6)	
4		(4, 4)		
2	(2, 2)			
1st roll	2	4	6	8

Of the sixteen cells, four cells show that the same number occurs on both rolls.

Hence the probability is $\frac{4}{16}=\mathbf{\frac{1}{4}}$

ii

2nd roll				
8	(2, 8)			
6		(4, 6)		
4			(6, 4)	
2				(8, 2)
1st roll	2	4	6	8

Of the sixteen cells, four cells show that the sum of the numbers is 10.

Hence the probability is $\frac{4}{16}=\mathbf{\frac{1}{4}}$

Question 4

Let P(A) be the probability that Emily goes for a run on a given day.

Then $\mathrm{P}(A)=0.3$ and $\mathrm{P}(A')=0.7$.

P(runs on either Monday or Tuesday) = P(runs on Monday but not on Tuesday) + P(runs on Tuesday but not on Monday)

$$\mathrm{P}(A)\times\mathrm{P}(A')+\mathrm{P}(A')\times\mathrm{P}(A)=0.3\times0.7+0.7\times0.3$$

$$=\mathbf{0.42}$$

Question 5

Let P(A) = a.

We write P(B) in terms of a.

P(B) = 4 × P(A) = 4a

Now, we substitute this in the addition rule.

P($A \cup B$) = P(A) + P(B) − P($A \cap B$)

$$0.7 = a + 4a - 0.05$$

$$0.75 = 5a$$

$$a = 0.15$$

Therefore, P(A) = **0.15**.

Question 6

First, we write the conditional probability in the form $\text{P}(A\,|\,B) = \dfrac{\text{P}(A \cap B)}{\text{P}(B)}$.

P(first number is 2 | sum is at least 4) $= \dfrac{\text{P}(\text{first number is 2 and the sum is at least 4})}{\text{P}(\text{sum is at least 4})}$

Next, we find the probability that the first number is 2 and the sum is at least 4.
P(first number is 2 and the sum is at least 4) = P(2, 3) + P(2, 4)

$$= \frac{1}{4} \times \frac{1}{3} + \frac{1}{4} \times \frac{1}{3}$$

$$= \frac{1}{6}$$

Next, we find the probability that the sum is at least 4.
P(sum is at least 4) = P(1, 3) + P(1, 4) + P(2, 3) + P(2, 4) + P(3, 1) + P(3, 2) + P(3, 4) + P(4, 1) + P(4, 2) + P(4, 3)

Each of these 10 probabilities has a probability $\frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$.

So, P(sum is at least 4) = $10 \times \frac{1}{12} = \frac{5}{6}$

Finally, we substitute these values in the conditional probability formula.

P(first number is 2 | sum is at least 4) $= \dfrac{\frac{1}{6}}{\frac{5}{6}} = \mathbf{\frac{1}{5}}$.

Question 7 (4 marks)

(✓ = 1 mark)

a

$$\mathbf{P(Box\ 1) \times P(Black) = \frac{1}{2} \times 1 = \frac{1}{2}}\ ✓$$

$$\mathbf{P(Box\ 2) \times P(Black) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}}$$

$$\mathrm{P(Black)} = \frac{1}{2} + \frac{1}{4} = \mathbf{\frac{3}{4}}\ ✓$$

b

$$\mathrm{P(Box\ 1|Black)} = \frac{\mathrm{P(Box\ 1 \cap Black)}}{\mathrm{P(Black)}}$$

$$= \frac{\mathbf{\frac{1}{2}}}{\mathbf{\frac{3}{4}}}\ ✓$$

$$= \frac{1}{2} \times \frac{4}{3}$$

$$= \mathbf{\frac{2}{3}}\ ✓$$

Question 8 (3 marks)

(✓ = 1 mark)

a The probability of tossing a head with an unbiased coin is $\frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$.

The probability of tossing a head with a biased coin is $\frac{1}{3} \times \frac{1}{3} = \mathbf{\frac{1}{9}}$. ✓

Therefore, the probability of tossing a head is $\frac{1}{3} + \frac{1}{9} = \mathbf{\frac{4}{9}}$. ✓

b Let P(*U*) be the probability of tossing an unbiased coin and P(*H*) be the probability of tossing a head.

The probability of selecting an unbiased coin and tossing a head is $\frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$.

The probability that Jo selected an unbiased coin given she tossed a head is:

$$\mathrm{P}(U \mid H) = \frac{\mathrm{P}(U \cap H)}{\mathrm{P}(H)} = \frac{\frac{1}{3}}{\frac{4}{9}} = \frac{1}{3} \times \frac{9}{4} = \mathbf{\frac{3}{4}}.\ ✓$$

Question 9 (6 marks)

(✓ = 1 mark)

Let W be the event that Sally walks Mack, and W' be the event that she does not walk Mack.

Let P be the event that the weather is pleasant, and P' be the event that the weather is unpleasant.

'If the weather is pleasant, the probability that she will walk Mack is $\frac{3}{4}$, means $\text{P}(W \mid P) = \frac{3}{4}$;

hence, $\text{P}(W' \mid P) = \frac{1}{4}$.

Similarly, 'if the weather is unpleasant, the probability that she will walk Mack is $\frac{1}{3}$', means

$\text{P}(W \mid P') = \frac{1}{3}$ and $\text{P}(W' \mid P') = \frac{2}{3}$.

a P(at least one morning walk) = 1 – P (no walk)

P(No walk on Monday) is $\text{P}(W' \mid P) = \mathbf{\frac{1}{4}}$.

P(No walk on Tuesday) = $\text{P}(W' \mid P') = \mathbf{\frac{2}{3}}$.✓ (both probabilities right)

P(No walk Monday or Tuesday) = $\frac{1}{4} \times \frac{2}{3} = \frac{1}{6}$

Hence, P(at least one morning walk) = 1 – P (no walk) $= 1 - \frac{1}{6} = \mathbf{\frac{5}{6}}$ ✓

An alternative (but more complicated) approach is to consider the events such that Mack gets either one or two walks over the two days with the known weather conditions.

P(at least one morning walk) = $\text{P}(W'W + WW' + WW)$ ✓

$$= \frac{1}{4} \times \frac{1}{3} + \frac{3}{4} \times \frac{2}{3} + \frac{3}{4} \times \frac{1}{3}$$

$$= \frac{1}{12} + \frac{6}{12} + \frac{3}{12}$$

$$= \mathbf{\frac{5}{6}} ✓$$

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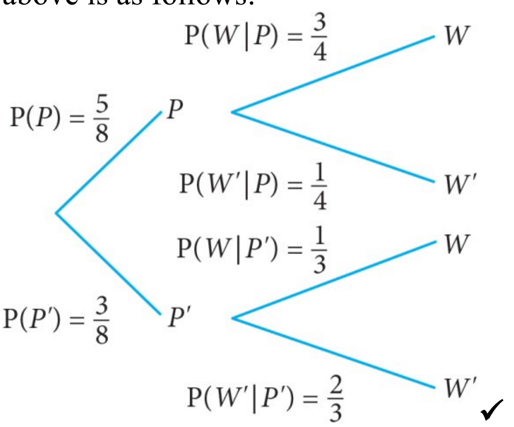
b **i** We are told that in the month of April, the probability of pleasant weather in the morning was $\frac{5}{8}$. Hence, in April $\text{P}(P) = \frac{5}{8}$ and $\text{P}(P') = 1 - \text{P}(P) = \frac{3}{8}$.

We require the probability that on a particular morning in April, Sally walked Mack. We can consider this probability by considering the sequence of events that lead to a walk and their probabilities. A tree diagram labelled with the probabilities determined above is as follows:



Mack can have a walk if either the weather is pleasant or unpleasant. Using the tree diagram, we see that the two paths that end with a walk on an April morning give

$$\text{P}(W) = \text{P}(P) \times \text{P}(W \mid P) + \text{P}(P') \times \text{P}(W \mid P')$$

$$= \left(\frac{5}{8} \times \frac{3}{4}\right) + \left(\frac{3}{8} \times \frac{1}{3}\right)$$

$$= \frac{15}{32} + \frac{1}{8}$$

$$= \mathbf{\frac{19}{32}} \checkmark$$

ii We require the probability that on a particular morning in April, the weather was pleasant, given that Sally walked Mack that morning. This is $\text{P}(P \mid W)$.

$$\text{P}(P \mid W) = \frac{\text{P}(P \cap W)}{\text{P}(W)} \checkmark$$

$$= \frac{\frac{5}{8} \times \frac{3}{4}}{\frac{19}{32}}$$

$$= \frac{15}{32} \times \frac{32}{19}$$

$$= \mathbf{\frac{15}{19}} \checkmark$$

Question 10 (3 marks)

(✓ = 1 mark)

a The probability of selecting a motor from line A $=\frac{40}{90}=\frac{4}{9}$.

As 5% of the motors in Line A are faulty, the probability of selecting a faulty motor $=\frac{5}{100}=\frac{1}{20}$.

The probability of selecting a motor from line B $=\frac{50}{90}=\frac{5}{9}$.

As 8% of the motors in Line B are faulty, the probability of selecting a faulty motor $=\frac{8}{100}=\frac{2}{25}$.

Let F represent a faulty motor being selected.

$$\mathrm{P}(F)=\mathbf{\frac{4}{9}\times\frac{1}{20}+\frac{5}{9}\times\frac{2}{25}}\checkmark$$
$$=\mathbf{\frac{1}{15}}\checkmark$$

b Use the conditional probability formula.

$$\mathrm{P}(A\,|\,F)=\frac{\mathrm{P}(A\cap F)}{\mathrm{P}(F)}$$
$$=\frac{\frac{4}{9}\times\frac{1}{20}}{\frac{1}{15}}$$
$$=\frac{\frac{1}{45}}{\frac{1}{15}}$$
$$=\frac{1}{45}\times\frac{15}{1}$$
$$=\mathbf{\frac{1}{3}}\checkmark$$

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Question 11 (3 marks)

(✓ = 1 mark)

a Let $\mathrm{P}(C)$ represent the probability that a doughnut has custard.

Let $\mathrm{P}(G)$ represent the probability that a doughnut is glazed.

Use the information provided to complete a probability table.

$\cap$	C	C'	
G	$\frac{1}{10}$	$\frac{2}{10}=\frac{1}{5}$	$\frac{3}{10}$
G'	$\frac{4}{10}=\frac{2}{5}$	$\frac{3}{10}$	$\frac{7}{10}$
	$\frac{5}{10}=\frac{1}{2}$	$\frac{5}{10}=\frac{1}{2}$	1

From the table, $\mathrm{P}(G' \cap C)=\frac{4}{10}=\mathbf{\frac{2}{5}}$.✓

b The total number of glazed doughnuts $=0.3\times 20=6$.

There are g glazed donuts in Box A so there are $6-g$ doughnuts in Box B.

$$\mathrm{P}(B \mid G)=\frac{\mathrm{P}(B\cap G)}{\mathrm{P}(G)}$$

$$=\frac{\frac{6-g}{\cancel{20}}}{\frac{6}{\cancel{20}}} ✓$$

$$=\frac{6-g}{6}$$

$$=\mathbf{1-\frac{g}{6}} ✓$$

Question 12 [SCSA MM2021 Q10b] (2 marks)

(✓ = 1 mark)

Score	Combinations	Probability
7	3, 4 or 2, 5 or 1, 6	$\frac{2}{8}\times\frac{1}{6}+\frac{2}{8}\times\frac{1}{6}+\frac{4}{8}\times\frac{1}{6}=\frac{8}{48}$
8	3, 5 or 2, 6	$\frac{4}{48}$
9	3, 6	$\frac{2}{48}$

$$\text{probability of a prize} = \frac{8}{48}+\frac{4}{48}+\frac{2}{48}=\frac{14}{48}=\frac{7}{24}$$

shows how to determine at least one probability✓

correctly shows all three probabilities and shows they sum to $\mathbf{\frac{7}{24}}$ ✓

Question 13 (3 marks)

(✓ = 1 mark)

a P(different colours) = P(red followed by yellow) + P(yellow followed by red)

$=\frac{5}{8}\times\frac{3}{7}+\frac{3}{8}\times\frac{5}{7}$ ✓ (show correct probability for each colour)

$=\frac{30}{56}$

$=\frac{15}{28}$ ✓

b P(same colour) = P(two red) + P(two yellow)

$=\frac{5}{8}\times\frac{4}{7}+\frac{3}{8}\times\frac{2}{7}$ ✓ (show correct probability for each colour)

$=\frac{26}{56}$

$=\frac{13}{28}$ ✓

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Question 14 (3 marks)

(✓ = 1 mark)

a P(wins next two games) = P(WW) = 0.7 × 0.7 = **0.49**✓

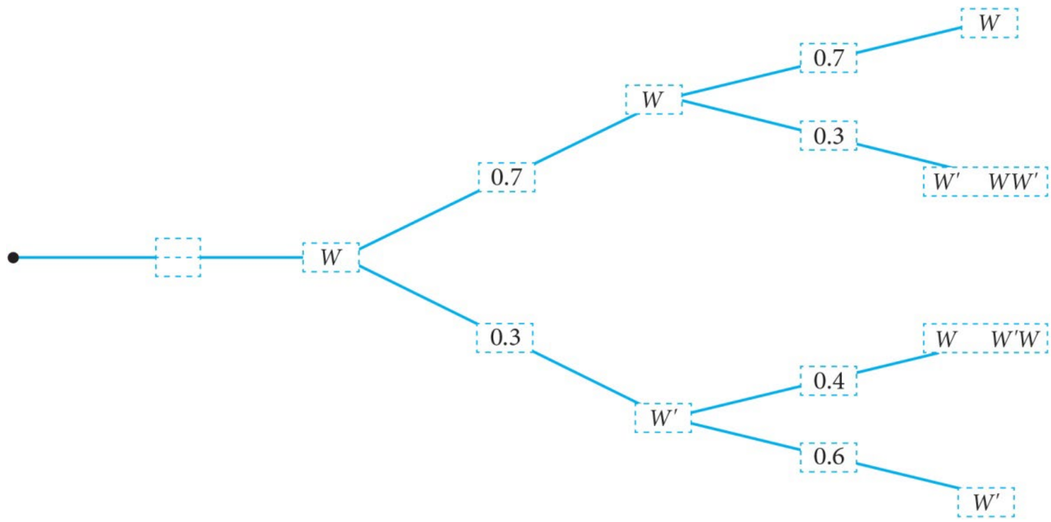
b Let W be the event that Demelza wins a game. The only other event is that she loses a game.

If she wins a game, the probability that she will win the next game is 0.7. From this, we can deduce that if she wins a game, the probability that she will lose the next game is 0.3.

If she loses a game, the probability that she will lose the next game is 0.6. From this, we can deduce that if she loses a game, the probability that she will win the next game is 0.4.

This shows that the order of events is important; hence, we are dealing with a question on conditional probability. A tree diagram composed of two stages and two branches per stage is appropriate for this situation.

Using the probabilities above, label the branches of the tree diagram.



Consider the ways in which Demelza wins exactly one of her next two games. This is either ***WW'* or *W'W*.**✓

From the tree diagram, we can calculate

P(exactly one win out of the two games that follow a win) = P(WW') + P($W'W$)

= 0.7 × 0.3 + 0.3 × 0.4

= **0.33**✓

EXERCISE 5.2 Discrete probability distributions

Question 1

Let R represent drawing a red marble and Y represent drawing a yellow marble.

The probability that the marbles are the same colour will be $P(RR) + P(YY)$.

$$\begin{aligned} P(RR)+P(YY) &= \frac{4}{6}\times\frac{3}{5}+\frac{2}{6}\times\frac{1}{5} \\ &= \frac{12}{30}+\frac{2}{30} \\ &= \frac{14}{30} \\ &= \frac{7}{15} \end{aligned}$$

The correct response is **C**.

Question 2

a There are four favourable outcomes where the sum is 5.

(1, 4), (4, 1), (2, 3), (3, 2)

There are $7 \times 7 = 49$ possible outcomes.

Hence $P(X=5) = \mathbf{\frac{4}{49}}$

b There are six favourable outcomes where the sum is less than 5.

(1, 1), (1, 2), (2, 1), (1, 3), (3, 1), (2, 2)

Hence $P(X<5)=\frac{6}{49}$

$P(X\geq 5)=1-P(X<5)$

$P(X\geq 5)=1-\frac{6}{49}$

$=\mathbf{\frac{43}{49}}$

Question 3

a Each outcome has the same probability of occurring, so it is a **uniform discrete probability distribution**.

b $P(X=x)=\frac{1}{8}$ for $x\in\{1, 2,.., 8\}$.

x	1	2	3	4	5	6	7	8
$P(X = x)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

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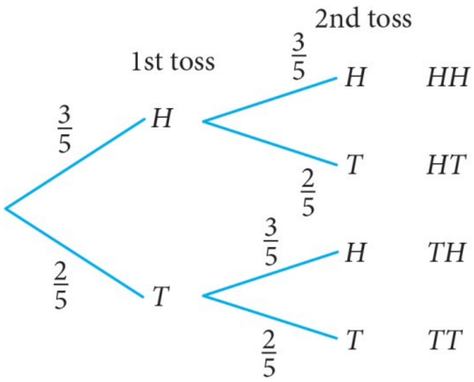
Question 4

Let X represent the number of tails that are obtained on two tosses of the coin.

First, we will draw a tree diagram to illustrate the problem.



Next, we will draw a probability distribution table where X represents the number of tails and $x \in \{0, 1, 2\}$.

x	0	1	2
$P(X = x)$			

Next, we use the tree diagram to find the probabilities.

$$P(X=0)=\frac{3}{5}\times\frac{3}{5}=\frac{9}{25}$$

$$P(X=1)=\frac{3}{5}\times\frac{2}{5}+\frac{2}{5}\times\frac{3}{5}=\frac{12}{25}$$

$$P(X=2)=\frac{2}{5}\times\frac{2}{5}=\frac{4}{25}$$

Finally, we write the probabilities in the table.

The probability distribution of the number of tails is

$\boldsymbol{x}$	**0**	**1**	**2**
$\mathbf{P}(\boldsymbol{X} = \boldsymbol{x})$	$\mathbf{\frac{9}{25}}$	$\mathbf{\frac{12}{25}}$	$\mathbf{\frac{4}{25}}$

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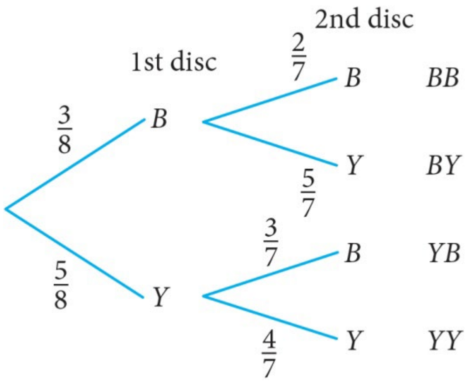
Question 5

First, we will draw a tree diagram to illustrate the problem.



Next, we will draw a probability distribution table where X represents the number of yellow discs selected and $x \in \{0, 1, 2\}$.

x	0	1	2
$P(X = x)$			

Next, we use the tree diagram to find the probabilities.

$$P(X = 0) = \frac{3}{8} \times \frac{2}{7} = \frac{3}{28}$$

$$P(X = 1) = \frac{3}{8} \times \frac{5}{7} + \frac{5}{8} \times \frac{3}{7} = \frac{15}{28}$$

$$P(X = 2) = \frac{5}{8} \times \frac{4}{7} = \frac{5}{14}$$

Finally, we write the probabilities in the table.

The probability distribution of the number of yellow discs selected is

x	**0**	**1**	**2**
$P(X = x)$	$\frac{3}{28}$	$\frac{15}{28}$	$\frac{5}{14}$

Question 6

As this is a probability distribution, the sum of the probabilities is one.

$\sum p(x) = 1$

$p + 0.3p + p^2 - 0.2 + p^2 - 0.3 = 1$

$2p^2 + 1.3p - 1.5 = 0$

Factorise and solve for p.

$p = -1.25$ or 0.6

As p is a probability, $p \geq 0$.

$\boldsymbol{p = 0.6}$

Question 7

As this is a probability distribution, the sum of the probabilities is one.

$\sum f(x) = 1$

$k \times 1 + k \times 2 + k \times 3 + k \times 4 = 1$

$10k = 1$

$k = \mathbf{\frac{1}{10}}$

Question 8

First, we will list the possible options where Emma gets a total of two green lights on two successive trips.

Let P(a, b) be the probability of Emma getting a green lights on the first trip and b green lights on the second trip.

P(total of two green lights) = P(0, 2) + P(1, 1) + P(2, 0)

Now, we will find the probabilities from the table and multiply the probabilities for each pair.

P(total of two green lights) = $(0.5 \times 0.2) + (0.3 \times 0.3) + (0.2 \times 0.5)$

$= 0.1 + 0.09 + 0.1$

P(total of two green lights) = **0.29**

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Question 9 (3 marks)

(✓ = 1 mark)

As this is a probability distribution, the sum of the probabilities is one.

$0.2 + 0.6p^2 + 0.1 + 1 - p + 0.1 = 1$ ✓

$$\mathbf{0.6p^2 - p + 0.4 = 0} \checkmark$$
$$\mathbf{6p^2 - 10p + 4 = 0}$$
$$\mathbf{3p^2 - 5p + 2 = 0}$$
$$\mathbf{(3p-2)(p-1) = 0}$$
$$\mathbf{p = \frac{2}{3}, 1} \checkmark$$

Alternatively, use the quadratic formula.

$$\mathbf{0.6p^2 - p + 0.4 = 0} \checkmark$$
$$\mathbf{p = \frac{1 \pm \sqrt{1-0.96}}{1.2}} \checkmark \text{ (use quadratic formula)}$$
$$\mathbf{= \frac{1 \pm 0.2}{1.2} = 1 \text{ or } \frac{0.8}{1.2} = \frac{2}{3}}$$
$$\mathbf{p = \frac{2}{3}, 1} \checkmark$$

Question 10 (3 marks)

(✓ = 1 mark)

As this is a probability distribution, the sum of the probabilities is one.

$$\sum \mathrm{P}(X = x) = p^2 + p^2 + \frac{p}{4} + \frac{4p+1}{8} = 1$$

Factorise and solve for p.

$$\mathbf{2p^2 + \frac{6p+1}{8} = 1} \checkmark$$
$$16p^2 + 6p + 1 = 8$$
$$16p^2 + 6p - 7 = 0$$
$$\mathbf{(8p+7)(2p-1) = 0} \checkmark$$
$$\text{As } p \geq 0,\ \mathbf{p = \frac{1}{2}}. \checkmark$$

Question 11 (2 marks)

(✓ = 1 mark)

P(Same number of red lights on both days) = **P(0, 0) + P(1, 1) + P(2, 2) + P(3, 3)**.✓

= (0.1 × 0.1) + (0.2 × 0.2) + (0.3 × 0.3) + (0.4 × 0.4) = **0.3**.✓

Question 12 (4 marks)

(✓ = 1 mark)

a We require the probability that Daniel receives only one telephone call on each of the three consecutive days.

The probability that he gets one call in a day is $P(X = 1) = 0.2$ from the table.

If the number of calls he receives in a day is independent of the number he received on the previous days,

P(one call on each day for three consecutive days)

= $P(X = 1$ on day 1$) \times P(X = 1$ on day 2$) \times P(X = 1$ on day 3$)$

= 0.2^3 = **0.008**. ✓

b We are told that Daniel receives telephone calls on both Monday and Tuesday. We require the probability that Daniel receives a total of four calls over these two days.

P(Total 4 calls | Calls ≥ 1 on Monday and Tuesday).

Let P(M*a*, T*b*) be the probability that Daniel receives *a* calls on Monday and *b* calls on Tuesday.

$$\text{P}(4\text{ Calls} \mid \text{Calls} \geq 1 \text{ Mon \& Tues}) = \frac{\mathbf{P(4\ Calls \cap Calls \geq 1\ on\ Monday\ and\ Tuesday)}}{\mathbf{P(Calls \geq 1\ on\ Monday\ and\ Tuesday)}} ✓$$

$$= \frac{\text{P(M1, T3)} + \text{P(M2, T2)} + \text{P(M3, T1)}}{0.8 \times 0.8}$$

$$= \frac{\mathbf{0.2 \times 0.1 + 0.5 \times 0.5 + 0.1 \times 0.2}}{\mathbf{0.8 \times 0.8}} ✓$$

$$= \frac{0.29}{0.64}$$

$$= \frac{\mathbf{29}}{\mathbf{64}} ✓$$

Question 13 [SCSA MM2016 Q15abii] (4 marks)

(✓ = 1 mark)

a

	Roll two				
	Sum of two rolls	1	2	3	4
Roll one	1	1 + 1 = 2	3	**4**	**5**
	2	3	**4**	**5**	**6**
	3	**4**	5	**6**	**7**
	4	**5**	**6**	**7**	**8**

enters all missing totals in table✓

b **i**

x	**2**	**3**	**4**	**5**	**6**	**7**	**8**
$P(X=x)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

completes table✓

ii

$$P(S \le 5) = \frac{4+3+2+1}{16} = \frac{10}{16} = \frac{5}{8}$$

uses all allowed values of sums✓

determines probability✓

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EXERCISE 5.3 Measures of centre and spread

Question 1

First, we will list the possible options where Danni receives five text messages over two consecutive hours.

Let P(a, b) be the probability that Danni receives a text messages in the first hour and b text messages in the second hour.

P(five text messages) = P(2, 3) + P(3, 2)

Now, we will find the probabilities from the table and multiply the probabilities for each pair.

P(five text messages) = $(0.4 \times 0.2) + (0.2 \times 0.4)$

$= 0.08 + 0.08$

P(five text messages) = 0.16

The correct response is **C**.

Question 2

As this is a probability distribution, the sum of the probabilities is one.

$\sum p(x) = 1$

$0.15 + 4p + 0.25 + 2p = 1$

$6p = 0.6$

$p = 0.1$

The correct response is **B**.

Question 3

First, we rewrite the table with rows as columns and add a column headed $x \times p(x)$. Then, we calculate the product of the x values and their probabilities.

x	$p(x)$	$x \times p(x)$
0	0.02	0.00
1	0.13	0.13
2	0.38	0.76
3	0.17	0.51
4	0.05	0.20
5	0.25	1.25
Total	1	2.85

The sum of the $x \times p(x)$ column is the expected value of X or $E(X)$.

$E(X) = \sum(x \times p(x))$

The mean or $E(X)$ is **2.85**.

Question 4

First, we rewrite the table with rows as columns and add a column headed $x \times p(x)$. Then, we calculate the product of the x values and their probabilities.

x	$p(x)$	$x \times p(x)$
5	a	$5a$
6	0.35	2.1
7	b	$7b$
8	0.15	1.2
9	0.1	0.9
Total	1.0	6.4

The sum of the $p(x)$ column must be 1.

The sum of the $x \times p(x)$ column must be $E(X) = 6.4$.

Next, we write the equations for the total of the $p(x)$ and $x \times p(x)$ columns.

$a + 0.35 + b + 0.15 + 0.1 = 1$

$$a + b = 0.4 \quad \ldots [1]$$

$5a + 2.1 + 7b + 1.2 + 0.9 = 6.4$

$$5a + 7b = 2.2 \quad \ldots [2]$$

Finally, we solve the simultaneous equations [1] and [2] to find the values of a and b.

Multiplying [1] by 5 gives

$$5a + 5b = 2 \quad \ldots [3]$$

[2] − [3] gives

$$2b = 0.2$$

$$b = 0.1$$

Substitute in [1]:

$$a + 0.1 = 0.4$$

$$a = 0.3$$

$\boldsymbol{a = 0.3, b = 0.1}$

Question 5

a First, we rewrite the table with rows as columns and add two columns headed $x \times p(x)$ and $x^2 \times p(x)$ for calculating $E(X)$ and $E(X^2)$ respectively.

x	$p(x)$	$x \times p(x)$	$x^2 \times p(x)$
1	0.1	0.1	0.1
2	0.4	0.8	1.6
3	0.1	0.3	0.9
4	0.4	1.6	6.4
Total	1.0	$E(X) = 2.8$	$E(X^2) = 9.0$

The total of the $x \times p(x)$ column is $E(X)$.

The mean or $E(X)$ is **2.8**.

b The total of the $x^2 \times p(x)$ column is $E(X^2)$.

$E(X^2) = 9.0$

We use the computational formula to find $\text{Var}(X)$.

$\text{Var}(X) = E(X^2) - \mu^2$

$\text{Var}(X) = 9 - 2.8^2$

$= 1.16$

The variance or $\text{Var}(X)$ is **1.16**.

c We find the standard deviation using the formula:

$\text{SD}(X) = \sqrt{\text{Var}(X)}$

$\text{SD}(X) = \sqrt{1.16} \approx 1.077$

The standard deviation or $\text{SD}(X)$ is **1.077**.

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Question 6

Construct a table with four columns and perform the calculations shown.

x	$p(x)$	$x \times p(x)$	$(x - \mu)^2 \times p(x)$
0	0.1	0	0.4
1	0.2	0.2	0.2
2	0.3	0.6	0
3	0.4	1.2	0.4
		$\mu = 2$	$\sigma^2 = 1$

$\text{Var}(X)$: $\sigma^2 = 1$

The variance is 1.

Question 7

a We use the formula $E(aX + b) = a\,E(X) + b$.

$$\begin{aligned} E(2X - 3) &= 2E(X) - 3 \\ &= 2 \times 15 - 3 \\ &= \mathbf{27} \end{aligned}$$

b We use the formula $\text{Var}(aX + b) = a^2\,\text{Var}(X)$.

$$\begin{aligned} \text{Var}(2X - 3) &= 2^2\,\text{Var}(X) \\ &= 4 \times 12 \\ &= \mathbf{48} \end{aligned}$$

c $E(X) = 15$, $\text{Var}(X) = 12$, $E(Y) = 100$, $\text{Var}(Y) = 48$

$E(Y) = E(aX + b) = a \times E(X) + b$

$15a + b = 100$

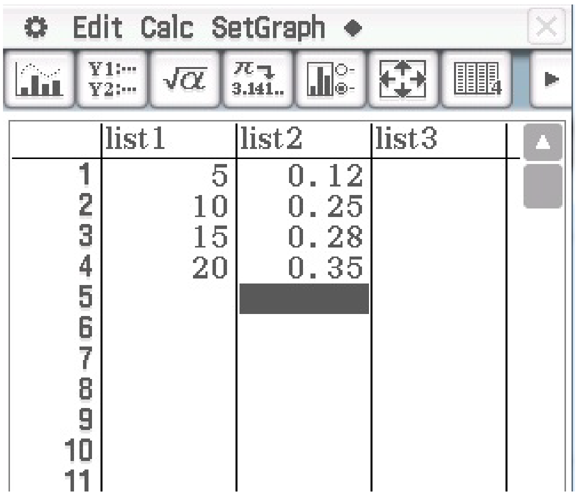
$\text{Var}(Y) = \text{Var}(aX + b) = a^2 \times \text{Var}(X)$

$a^2 \times 12 = 48$, so $\boldsymbol{a = 2}$

$15 \times 2 + b = 100$, so $\boldsymbol{b = 70}$

Question 8

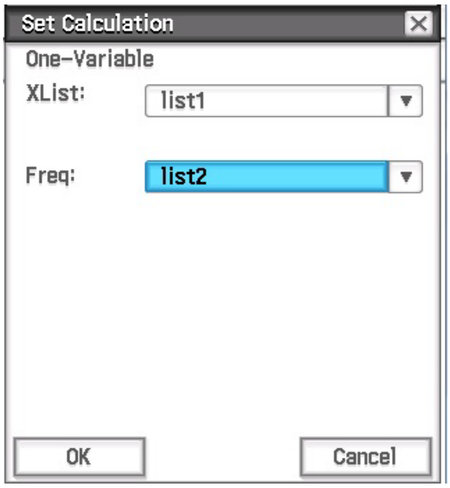
ClassPad



1 Tap **Menu** > **Statistics**.

2 Clear all lists.

3 Enter the values from the table into **list1** and **list2** as shown above.

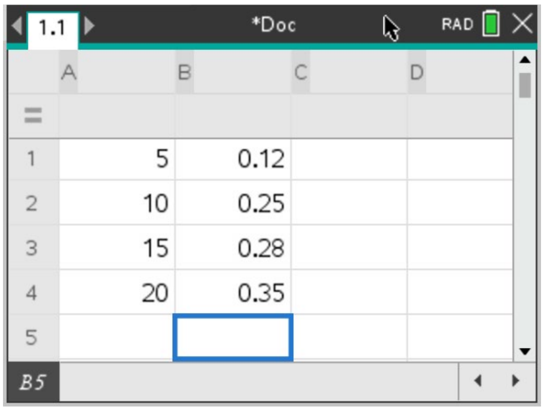


4 On the next screen, keep the **Num of Lists**: default setting of 1.

5 On the next screen shown above, in the **X1 List**: field, enter **a[ ]**.

6 In the **Frequency List:** field, enter **b[ ]**.

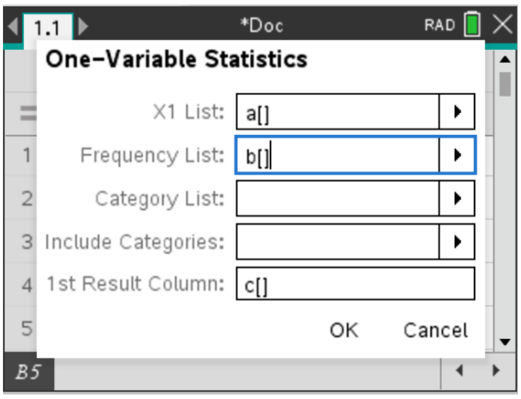
TI-Nspire



1 Add a **List & Spreadsheet** page.

2 Enter the data from the table in columns **A** and **B** as shown above.

3 Press **menu** > **Statistics** > **Stat Calculations** > **One-Variable Statistics.**



4 Tap **Calc** > **One-Variable**.

5 Keep the **XList:** field set to **list1**.

6 Change the **Freq**: field to **list2**.

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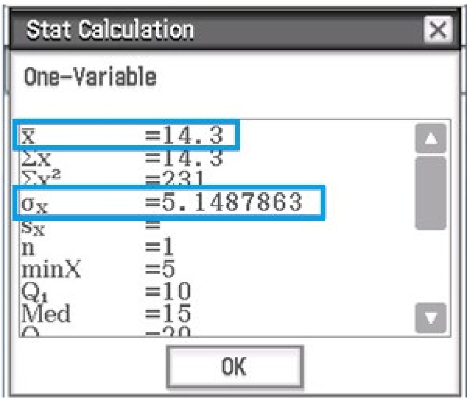
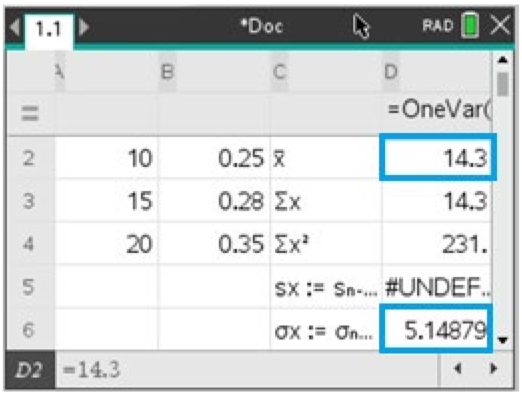


7 The labels and corresponding values will be displayed.

8 The **mean** and **standard deviation** values are highlighted in blue.

9 To calculate the variance, square the standard deviation.

E(*X*) = 14.30; Var(*X*) = 26.51; SD(*X*) = 5.15



7 The labels and corresponding values will be displayed in columns **C** and **D**.

8 The **mean** and **standard deviation** values are highlighted in blue.

9 To calculate the variance, square the standard deviation.

Question 9 (2 marks)

(✓ = 1 mark)

Use the table given to complete a table with three columns, x, $p(x)$ and $x \times p(x)$. ✓

Calculate the values for the third column and the column totals.

x	$p(x)$	$x \times p(x)$
0	0.2	0
1	0.2	0.2
2	0.5	1
3	0.1	0.3
Total	1	1.5

The total of the $x \times p(x)$ column is the expected value of X, that is, $E(X) =$ **1.5**.✓

Question 10 (3 marks)

(✓ = 1 mark)

a Use the table given to complete a table with three columns, x, $p(x)$ and $x \times p(x)$ using $p = \frac{2}{3}$.

Calculate the values for the third column and the column totals.

Convert decimals to fractions that have a common denominator.

x	$p(x)$	$x \times p(x)$
0	$\frac{2}{10}=\frac{6}{30}$	0
1	$\frac{4}{15}=\frac{8}{30}$	$\frac{4}{15}=\frac{8}{30}$
2	$\frac{1}{10}=\frac{3}{30}$	$\frac{2}{10}=\frac{6}{30}$
3	$\frac{1}{3}=\frac{10}{30}$	$1=\frac{30}{30}$
4	$\frac{1}{10}=\frac{3}{30}$	$\frac{4}{10}=\frac{12}{30}$
Total	1	$\frac{56}{30}$

✓

The total of the $x \times p(x)$ column is the expected value of X, that is,

$E(X)=\frac{56}{30}=\mathbf{\frac{28}{15}}$ ✓

b $E(X)=\frac{28}{15}=1\frac{13}{15}$

To determine $P(X \geq E(X))$, identify the x values in the table that are greater than or equal to $1\frac{13}{15}$ and add their probabilities.

$$P\left(X \geq 1\frac{13}{15}\right) = P(X=2) + P(X=3) + P(X=4)$$

$$= \frac{3}{30}+\frac{10}{30}+\frac{3}{30} \text{ (using the table from part } \textbf{a}\text{)}$$

$$= \frac{16}{30} = \mathbf{\frac{8}{15}}$$ ✓

Question 11 [SCSA MM2018 Q4] (4 marks)

(✓ = 1 mark)

a

n	0	1	2	3
$P(N = n)$	$\frac{2}{10}$	$\frac{4}{10}$	$\frac{3}{10}$	$\frac{1}{10}$

gives one correct entry✓

completes the table correctly✓

b $E(X) = \left(0\times\frac{2}{10}\right)+\left(1\times\frac{4}{10}\right)+\left(2\times\frac{3}{10}\right)+\left(3\times\frac{1}{10}\right)$

$= \frac{13}{10} = 1.3$

gives correct expression for *E*(*X*)✓

simplifies answer✓

Question 12 [SCSA MM2016 Q17] (7 marks)

(✓ = 1 mark)

a $E(X = 5) = E(X) + 5 = 75 + 5 = 80$

determines mean✓

b $\text{Var}(25 - 2X) = 2^2\text{Var}(X) = 4 \times 22^2 = 1936$

uses a positive factor of four✓

determines variance✓

c $15 = 22a$

$a = \frac{15}{22} \approx 0.682$

$60 = 75a + b$

$b = \frac{195}{22} \approx 8.864$

determines change on standard deviation first✓

sets up at least one equations for *a* and *b*✓

determines *a*✓

determines *b*✓

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Question 13 (10 marks)

(✓ = 1 mark)

a Let S be the event that a Superior statue is made. The only other event is S', the statue made is Regular and not Superior.

$$P(S \mid S) = p \Rightarrow P(S' \mid S) = 1 - p$$

We are also told that $\Pr(S \mid S') = p - 0.2$.

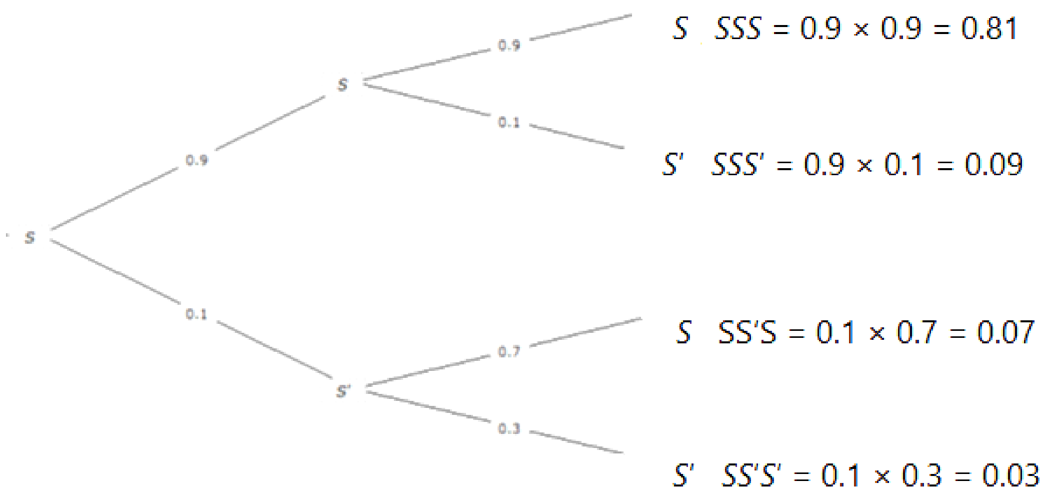
$$\therefore P(S' \mid S') = 1 - (p - 0.2) = 1.2 - p$$

On a particular day, Victoria knows that $p = 0.9$.

$\therefore$ **$P(S \mid S) = 0.9$, $P(S' \mid S) = 0.1$, $P(S \mid S') = 0.7$, $P(S' \mid S') = 0.3$** ✓

We can use a tree diagram of two stages (representing the next two statues) that follow the first statue. There are two branches at each stage representing the events S and S'. At the far left of the tree diagram is S as we know that the first statue is Superior.

The branches that follow are labelled with probabilities using the values obtained above.



We require that the third statue is Regular, given that the first statue inspected is Superior.

P(third statue is S' | the first statue is S) = $P(SSS') + P(SS'S')$

$= 0.9 \times 0.1 + 0.1 \times 0.3 = 0.09 + 0.03 =$ **0.12**✓

b $P(SSS \mid S) = P(S \mid S) \times P(S \mid S) \times P(S \mid S)$

$$= 0.9 \times 0.9 \times 0.9 = 0.729$$

The probability of one Superior statue being followed by three Superior statues is **0.729**.✓

c i From part **a**, $\mathrm{P}(S \mid S) = p$, $\mathrm{P}(S' \mid S) = 1 - p$, $\mathrm{P}(S \mid S') = p - 0.2$, $\mathrm{P}(S' \mid S') = 1.2 - p$.

Similarly to part **a**, P(third statue is S | the first statue is S) = **P(SSS) + P($SS'S$)**✓

This probability is 0.7, so

$$\mathrm{P}(S \mid S) \times \mathrm{P}(S \mid S) + \mathrm{P}(S' \mid S) \times \mathrm{P}(S \mid S') = 0.7$$

$$\boldsymbol{p \times p + (1 - p) \times (p - 0.2) = 0.7} \checkmark$$

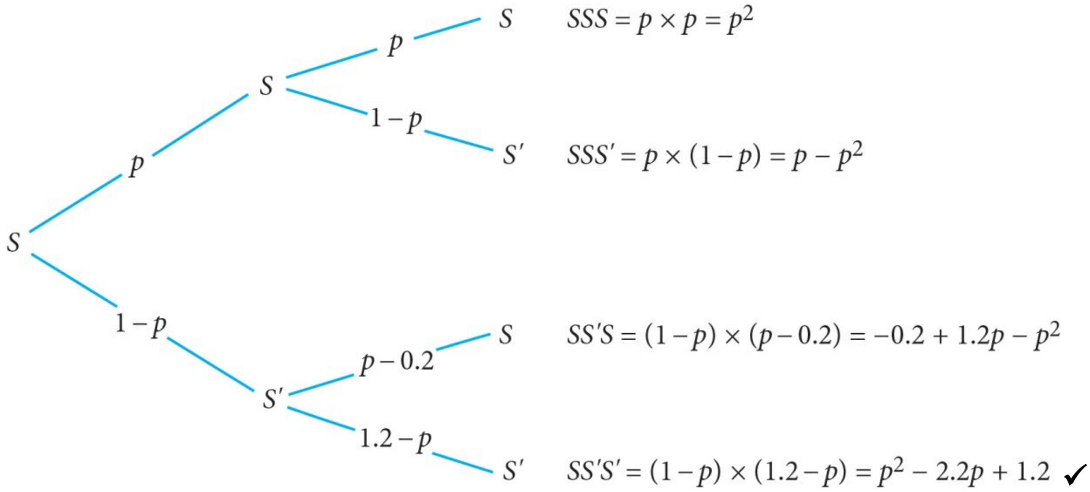
$$p^2 + p - 0.2 - p^2 + 0.2p = 0.7$$

$$1.2p = 0.9$$

$$p = \frac{0.9}{1.2} = 0.75$$

Hence $\boldsymbol{p = 0.75}$ as required.✓

Alternatively, use these probabilities to label a tree diagram as for part **a**.



P(third statue S | first statue S) = P(SSS) + P($SS'S$) = 0.7

Using the tree diagram,

$$\boldsymbol{p^2 - 0.2 + 1.2p - p^2 = 0.7} \checkmark$$

$$1.2p = 0.9$$

$$\therefore p = \frac{0.9}{1.2} = \frac{3}{4} = 0.75$$

Hence, the value of p on this day is **0.75** ✓, as required.

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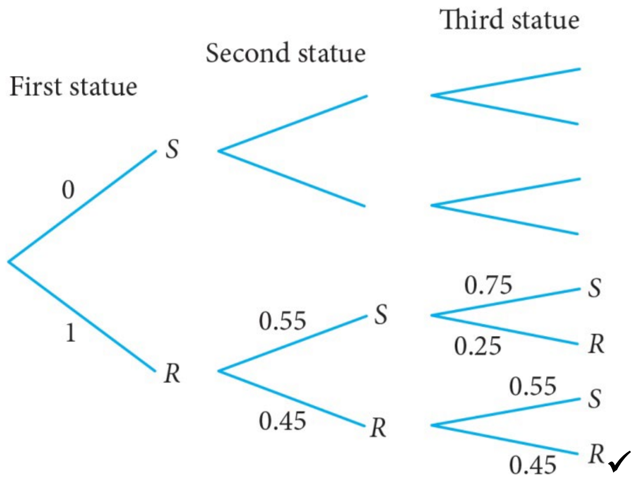
ii Using $p = 0.75$, we can determine the probabilities of the four different pairs of consecutive statue types that can occur. This information can be displayed on a tree diagram where we know that the first statue is Regular.



Using the tree diagram, we can calculate the individual probabilities that 0, 1 or 2 of the next two statues are Superior:

P(0 superior) = $1 \times 0.45 \times 0.45 =$ **0.2025**✓

P(1 superior) = $1 \times 0.55 \times 0.25 + 1 \times 0.45 \times 0.55 = 0.385$

P(2 superior) = $1 \times 0.55 \times 0.75 = 0.4125$✓

$E(X) = 0 \times 0.2025 + 1 \times 0.3850 + 2 \times 0.4125 =$ **1.21** ✓ statues

EXERCISE 5.4 The binomial distribution

Question 1

$$\begin{aligned}\mathrm{E}(X) &= 0\times0.05+1\times0.13+2\times0.27+3\times0.1+4\times0.25+5\times0.14+6\times0.06\\ &= 0.13+0.54+0.3+1+0.7+0.36\\ &= 3.03\end{aligned}$$

$$\begin{aligned}\mathrm{E}(X^2) &= 0\times0.05+1\times0.13+4\times0.27+9\times0.1+16\times0.25+25\times0.14+36\times0.06\\ &= 0.13+1.08+0.9+4+3.5+2.16\\ &= 11.77\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}(X^2)-(\mathrm{E}(X))^2\\ &= 11.77-3.03^2\\ &= 11.77-9.1809\\ &= 2.5891\end{aligned}$$

The correct response is **B**.

Question 2

$$\begin{aligned}\mathrm{E}(X) &= 0\times 3p + 1\times p + 2\times(1-4p)\\ &= p+2-8p\\ &= 2-7p\end{aligned}$$

The correct response is **E**.

Question 3

a X is the discrete random variable that the alarm will fail on a particular night. The alarm either fails or it does not fail, 2 possible outcomes, so it is a Bernoulli distribution.

The distribution is uniform, with mean 0.05. Hence $X \sim \textbf{Bern}(\textbf{0.05})$.

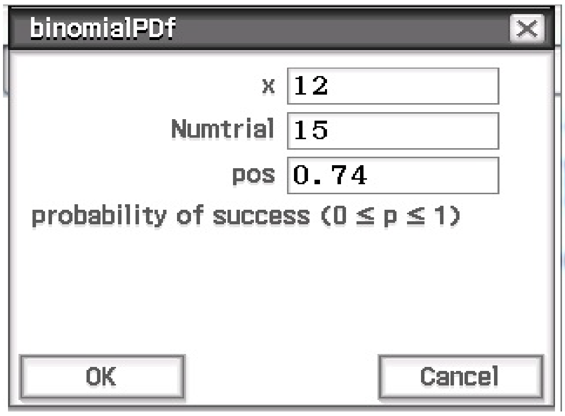
b $\mathrm{E}(X) = p = 0.05$

Mean = 0.05

Var(*X*) $= p(1-p) = 0.05(0.95) =$ **0.0475**

Question 4

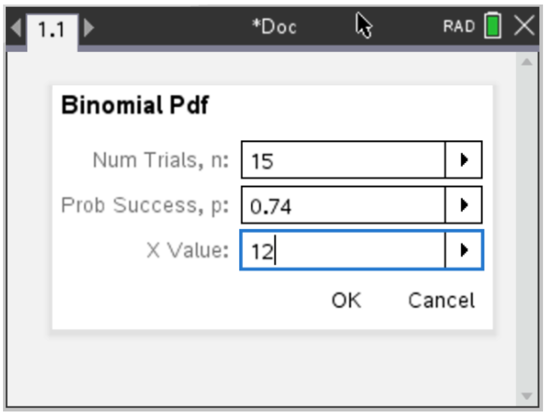
a **ClassPad**



1 Tap **Interactive > Distribution/Inv.Dist > Discrete > binomialPDf.**

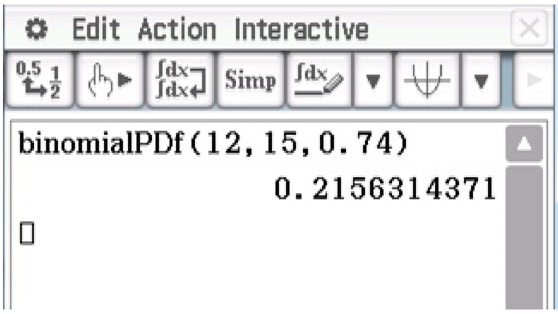
2 Enter the values as shown above.

TI-Nspire



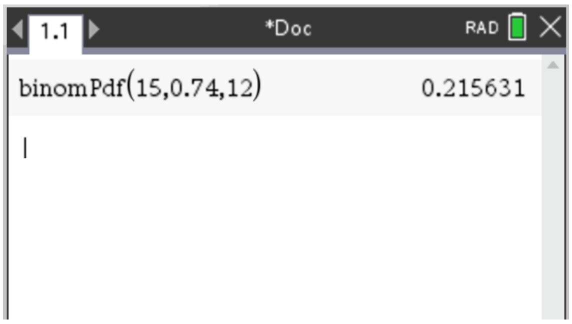
1 Press **menu > Probability > Distributions > Binomial Pdf**.

2 Enter the values as shown above.



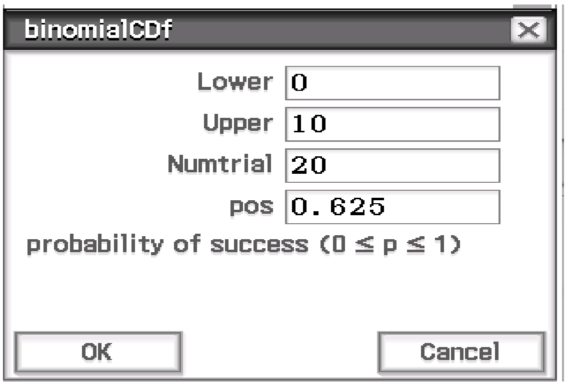
3 The binomial probability will be displayed.

P(X = 12) is **0.216** to three decimal places.



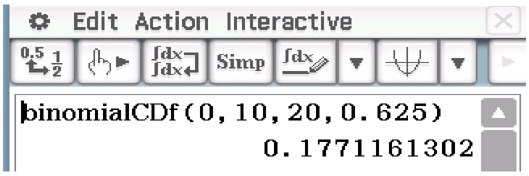
3 The binomial probability will be displayed.

b **ClassPad**



1 Tap **Interactive > Distribution/Inv.Dist > Discrete > binomialCDf.**

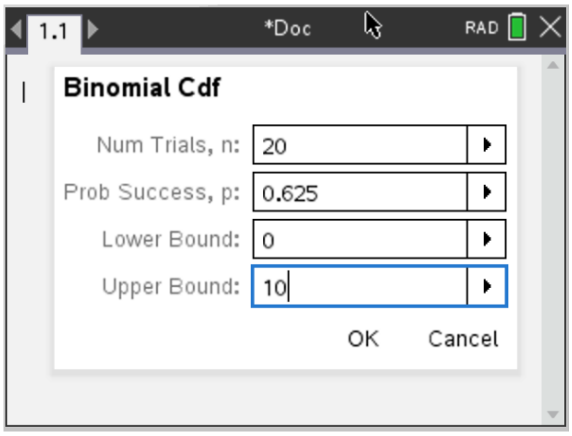
2 Enter the values as shown above.



3 The binomial probability will be displayed.

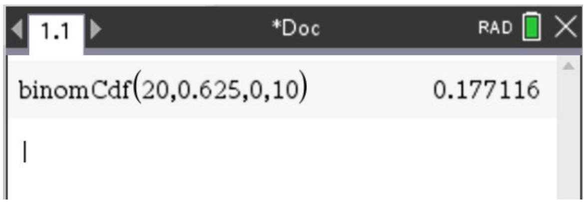
P(X < 11) is **0.177** to three decimal places.

TI-Nspire



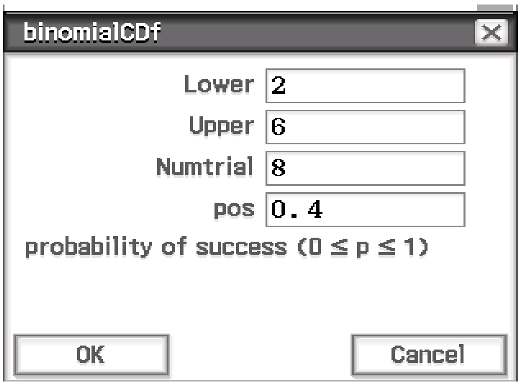
1 Press **menu > Probability > Distributions > Binomial Cdf**.

2 Enter the values as shown above.



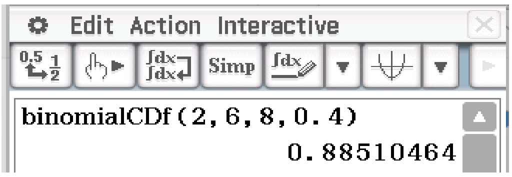
3 The binomial probability will be displayed.

c **ClassPad**



1 Tap **Interactive > Distribution/Inv.Dist > Discrete > binomialCDf.**

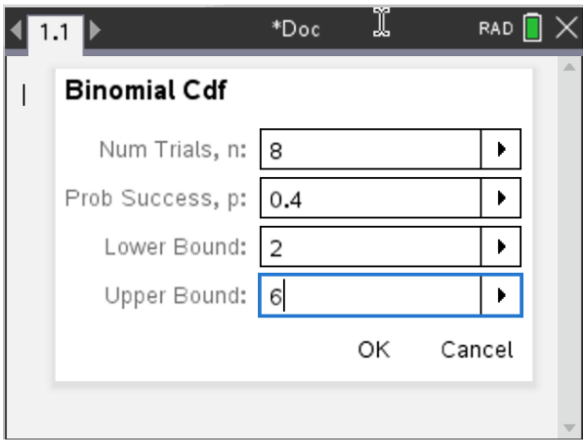
2 Enter the values as shown above.



3 The binomial probability will be displayed.

$P(1 < Y \leq 6)$ is **0.885** to three decimal places.

TI-Nspire



1 Press **menu > Probability > Distributions > Binomial Cdf**.

2 Enter the values as shown above.

3 The binomial probability will be displayed.

Question 5

a The distribution is binomial because there are two outcomes on each trial: guesses correctly or guesses incorrectly.

There are eight multiple-choice questions, so $n = 8$.

The probability of guessing an answer correctly is $p = \frac{1}{5} = 0.2$.

Binomial $n = 8,\ p = \frac{1}{5} = 0.2$

Now, we write the probability using the formula.

$$\text{P}(X = x) = {}^{n}C_{x}p^{x}(1-p)^{n-x}$$

If the student guesses half the question correctly, this will be four correct questions.

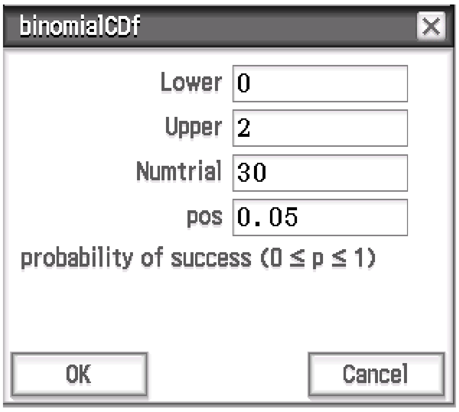
$$\begin{aligned}\text{P}(X = 4) &= {}^{8}C_{4}(0.2)^{4}(1-0.2)^{8-4} \\ &= 0.0458752 \\ &\approx \mathbf{0.046}✓\end{aligned}$$

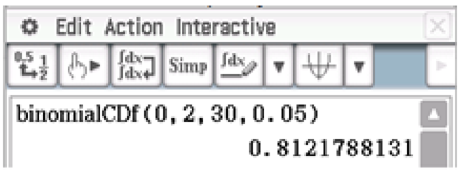
b There are 30 days in the month of April, so $n = 30$.

The probability of puncturing a tyre is $p = 0.05$.

The required probability is $\text{P}(X \le 2)$.

ClassPad





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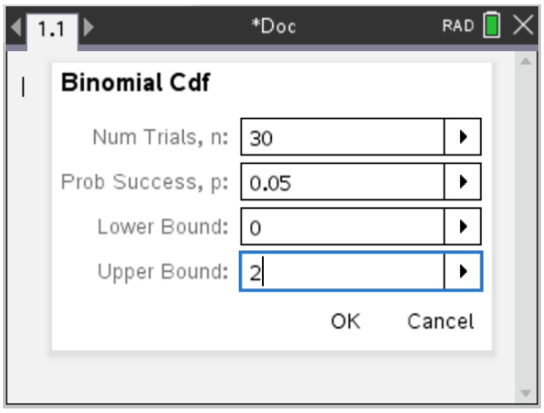

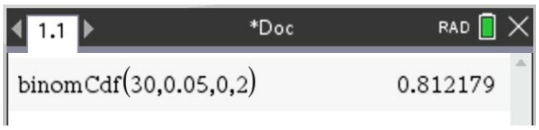

$\text{P}(X \le 2) \approx \mathbf{0.8122}$

Question 6

a **i** The distribution is binomial because there are two possible outcomes on each trial: scores a goal or does not score a goal.

X represents the number of goals.

$X \sim \text{Bin}(6, p)$

Now, we will calculate the probability of $X = 5$ using the formula.

$$\text{P}(X = x) = {}^nC_x p^x (1-p)^{n-x}$$

$$\text{P}(X = 5) = {}^6C_5 p^5 (1-p)^{6-5}$$

$$= \mathbf{6p^5(1-p)}$$

ii Now, we will calculate the probability of $X = 6$ using the formula.

$$\text{P}(X = x) = {}^nC_x p^x (1-p)^{n-x}$$

$$\text{P}(X = 6) = {}^6C_6 p^6 (1-p)^{6-6}$$

$$= \mathbf{p^6}$$

b Equate the answers in part **a**, and solve for p.

$6p^5(1-p) = p^6$

$6(1-p) = p$

$6 - 6p - p = 0$

$6 - 7p = 0$

$$p = \mathbf{\frac{6}{7}}$$

Question 7

a $\text{E}(X) = np = 90$ … [1]

$\text{Var}(X) = np(1 - p) = 36$ … [2]

We will solve the equations by substitution to find the values of n and p.

Substitute equation [1] into [2].

$90(1 - p) = 36$

$$1 - p = \frac{36}{90}$$

$$p = 1 - \frac{36}{90}$$

$$= \frac{3}{5}$$

$\boldsymbol{p = 0.6}$

Substitute into [1].

$n \times 0.6 = 90$

$\boldsymbol{n = 150}$

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b $E(X) = np = 10$ … [1]

$\text{Var}(X) = np(1-p) = 9$ … [2]

We will solve the equations by substitution to find the values of n and p.

Substitute equation [1] into [2].

$10(1-p) = 9$

$$1 - p = \frac{9}{10}$$

$$p = 1 - \frac{9}{10}$$

$$= \frac{1}{10}$$

$$= 0.1$$

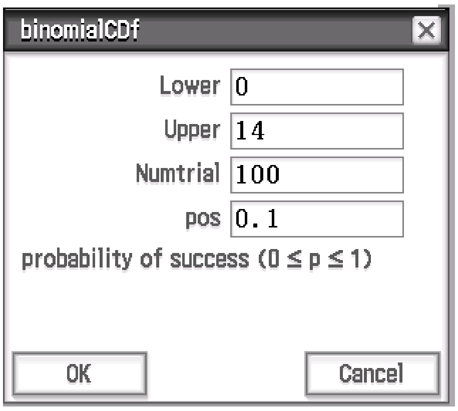
Substitute into [1].

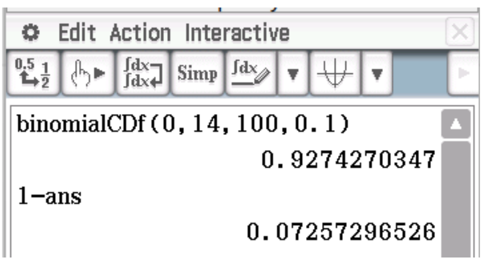
$n \times 0.1 = 10$

$n = 100$

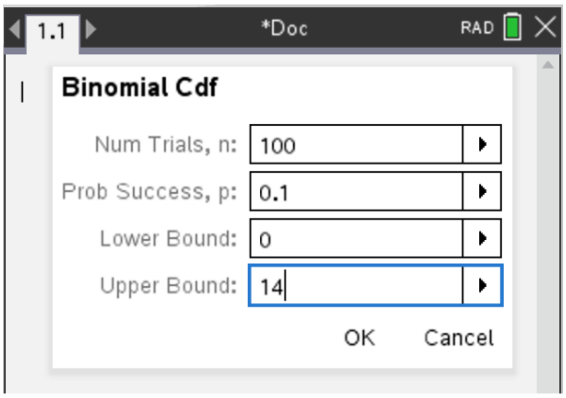
$X \sim \text{Bin}(100, 0.1)$

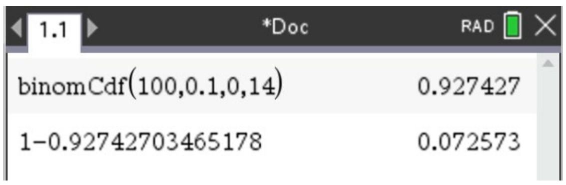
ClassPad





TI-Nspire

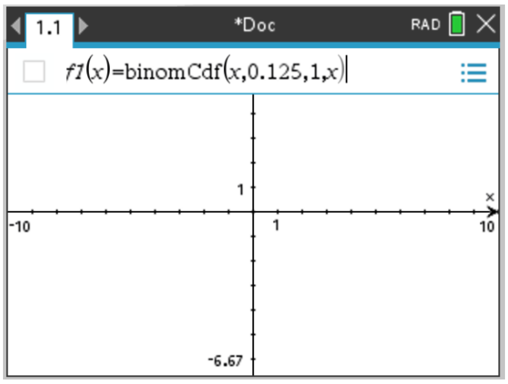




$P(X \geq 15) = \mathbf{0.07}$

Question 8

a **ClassPad**



1 Tap **Menu** > **Graph&Table**.

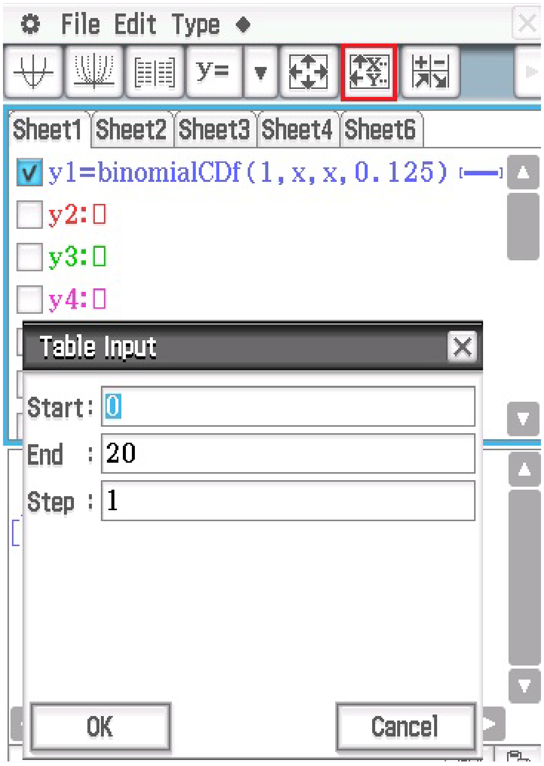
2 Open the **Keyboard** > **down arrow** > **Catalog**.

3 Tap **B** and scroll down to select **binomialCDf**.

4 Enter the values **(1,x,x,0.125)** as shown above.

5 Tap **Table Input** to set a suitable range of values.

TI-Nspire

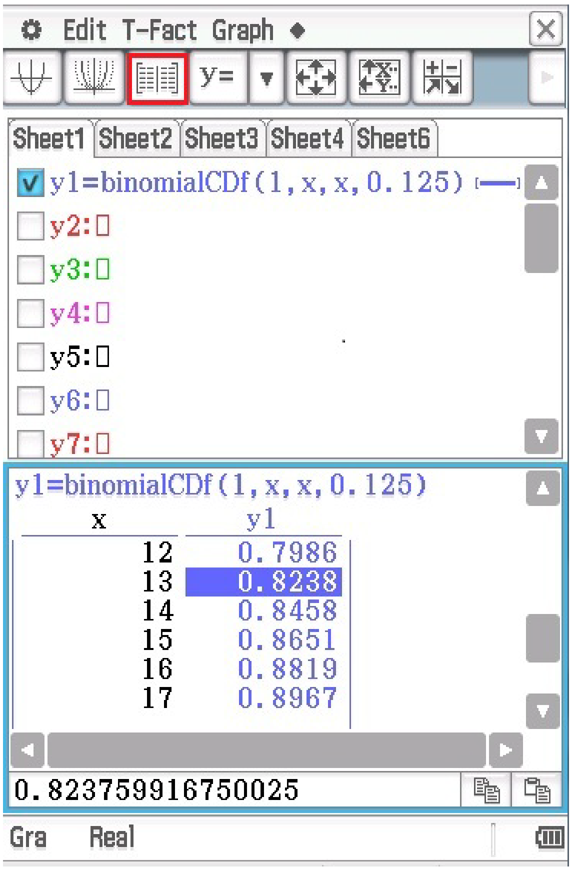


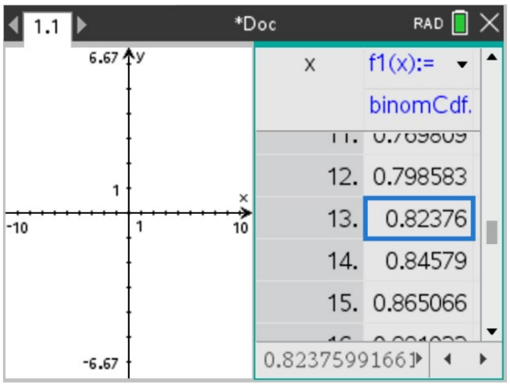
1 Add a **Graphs** page.

2 Press **catalogue** then **B** to jump to the functions starting with b.

3 Scroll down and select **binomCdf**.

4 Enter the values as shown above.





5 After pressing **enter**, no graph will appear.

6 Press **menu** > **Table** > **Split-screen Table**.

7 Scroll down the table on the right to the first binomCdf value greater than 0.8.

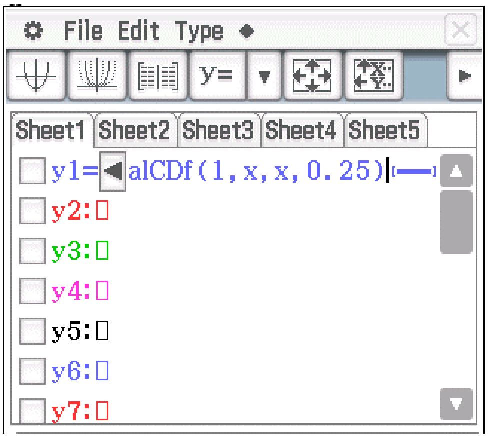
6 Tap **OK**.

7 Tap **Table** to view the values.

8 Scroll down the table to the first binomialCDf value greater than 0.8.

The smallest number of darts the player should throw is n = **13**.

b **ClassPad**



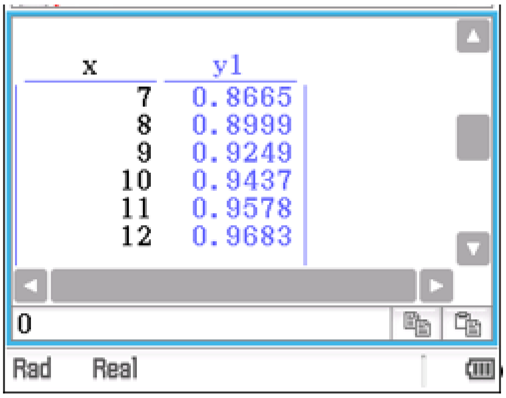
1 Tap **Menu** > **Graph&Table**.

2 Open the **Keyboard** > **down arrow** > **Catalog**.

3 Tap **B** and scroll down to select **binomialCDf**.

4 Enter the values **(1,x,x,0.25)** as shown above.

5 Tap **Table Input** to set a suitable range of values.



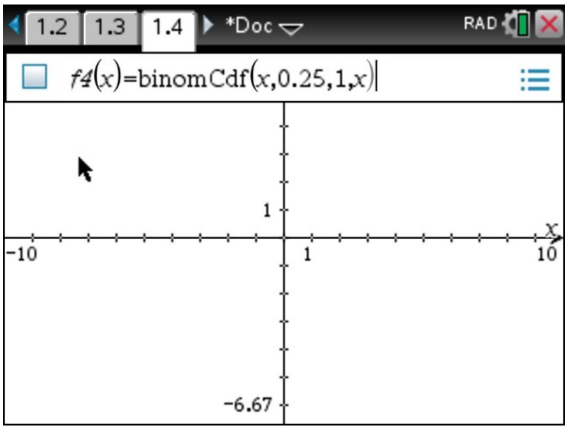
6 Tap **OK**.

7 Tap **Table** to view the values.

8 Scroll down the table to the first binomialCDf value one or more deliveries will not arrive on time or earlier (≥ 0.95).

The minimum value is $\boldsymbol{n} = \mathbf{11}$.

TI-Nspire

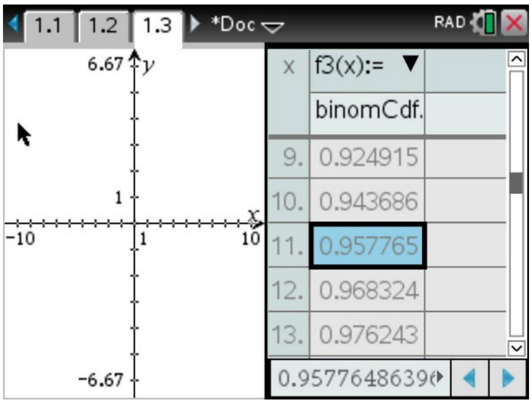


1 Add a **Graphs** page.

2 Press **catalogue** then **B** to jump to the functions starting with b.

3 Scroll down and select **binomCdf**.

4 Enter the values as shown above.



5 After pressing **enter**, no graph will appear.

6 Press **menu** > **Table** > **Split-screen Table**.

7 Scroll down the table on the right to the first binomCdf value greater than or equal to 0.95.

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Question 9 [SCSA MM2018 Q1] (9 marks)

a

y	0	1
$P(Y = y)$	$\frac{1}{5}$	$\frac{4}{5}$

completes first probability correctly✓

completes second probability correctly✓

b It is a Bernoulli distribution.

states the distribution name✓

c

$$\mu = \frac{4}{5}$$

$$\sigma = \sqrt{\frac{1}{5} \times \frac{4}{5}} = \frac{2}{5}$$

states the mean✓

states the simplified value of the standard deviation✓

d

$$X \sim \text{Bin}\left(5, \frac{4}{5}\right)$$

states the distribution name✓

states the parameters of the distribution✓

e

$$\begin{aligned} P(X = 2) &= \binom{5}{2}\left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right)^3 \\ &= \frac{5 \times 4 \times 16}{2 \times 5^5} \\ &= \frac{32}{625} \end{aligned}$$

correctly substitutes into binomial formula✓

states simplified probability✓

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Question 10 (3 marks)

(✓ = 1 mark)

a There are 10 tagged sheep out of a total of 30 sheep so the probability of selecting a tagged sheep is $\frac{10}{30}=\frac{1}{3}$. Sheep are selected at random four times a day so the binomial distribution is $X \sim \text{Bin}\left(4, \frac{1}{3}\right)$.

The probability that none of the sheep selected are tagged is:

$\text{P}(X=0)=\binom{4}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^4=\mathbf{\frac{16}{81}}$.✓

b The probability that at least one sheep is tagged is $\text{P}(X \geq 1)=1-\text{P}(X=0)$

From part **a.**, $1-\text{P}(X=0)=1-\frac{16}{81}=\frac{65}{81}$.

So $\text{P}(X \geq 1)=\mathbf{\frac{65}{81}}$.✓

c The probability that no sheep are tagged on any given day is $p=\frac{16}{81}$. Sheep are selected for six consecutive days so the binomial distribution is $Y \sim \text{Bin}\left(6, \frac{16}{81}\right)$. The probability that no tagged sheep are selected is $\Pr(Y=0)=\binom{6}{0}\left(\frac{16}{81}\right)^6\left(\frac{65}{81}\right)^0=\mathbf{\left(\frac{16}{81}\right)^6}$ or $\mathbf{\left(\frac{2}{3}\right)^{24}}$.✓

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Question 11 (4 marks)

(✓ = 1 mark)

We are told that a biased coin is tossed three times. Let H be the event of the coin showing a head and T be the event that the coin shows a tail. $P(H) = p$.

a **i** Three heads from three coin tosses is the event HHH.

This has probability $P(HHH) = P(H)^3 = \boldsymbol{p^3}$.✓

ii Use the Binomial distribution where $X \sim \text{Bin}(3, p)$.

$P(X = 2) = {}^3C_2 p^2(1-p)^{3-2} = \mathbf{3p^2(1-p)}$✓

Alternatively, use first principles.

To determine this, we can use a tree diagram of three stages with two branches per stage.

These branches will be labelled with $P(H) = p$ and $P(T) = 1 - P(H) = 1 - p$.

Alternatively, we can recognise that obtaining two heads and a tail from three tosses occurs for the events HHT or HTH or THH.

Hence, P(two heads and one tail from the three tosses) = $P(HHT) + P(HTH) + P(THH)$.

Consider each of the probabilities on the right of the equation in turn:

$P(HHT) = P(H) \times P(H) \times P(T) = p^2(1-p)$

$P(HTH) = P(H) \times P(T) \times P(H) = p \times (1-p) \times p = p^2(1-p)$

$P(THH) = P(T) \times P(H) \times P(H) = (1-p)p^2 = p^2(1-p)$

So, P(two heads and one tail) = $P(HHT) + P(HTH) + P(THH) = \mathbf{3p^2(1-p)}$✓

b We are told that the probability of obtaining three heads equals the probability of obtaining two heads and a tail.

From part **a i**, we know that $P(HHH) = p^3$.

From part **a ii**, we know that P(two heads and one tail) = $3p^2(1-p)$.

$p^3 = 3p^2(1-p)$

$p^3 = 3p^2 - 3p^3$

$4p^3 - 3p^2 = 0$

$p^2(4p - 3) = 0$✓

$\boldsymbol{p = 0, \frac{3}{4}}$✓

Question 12 (2 marks)

(✓ = 1 mark)

The coffee-buying behaviour of a customer in a restaurant has only two outcomes; they buy a coffee or they do not. We know that a customer buys a coffee with probability $p = 0.5$ and does not buy a coffee with probability $1 - p = 0.5$.

We are told to assume that the choice of each customer is independent of other customers.

Hence, the coffee-buying behaviour of a customer is a Bernoulli trial. Let X be a random variable representing the number of customers who buy coffee out of the four observed.

$X \sim \text{Bin}(4, 0.5)$ ✓

We require the probability that more than two of these customers order coffee.

This is $P(X > 2) = P(X = 3) + P(X = 4)$

$$= {}^4C_3(0.5)^3(0.5)^1 + {}^4C_4(0.5)^4(0.5)^0$$

$$= \frac{4}{16} + \frac{1}{16} = \mathbf{\frac{5}{16}} ✓$$

Question 13 (5 marks)

(✓ = 1 mark)

a $0.1 + 0.4 + 0.3 + p = 1$

$\boldsymbol{p = 0.2}$ ✓

b $E(X) = -1 \times 0.1 + 0 \times 0.4 + 1 \times 0.3 + 2p$

$= -0.1 + 0.3 + 0.4$

$\mathbf{= 0.6}$ ✓

c $E(X^2) = 1 \times 0.1 + 0 \times 0.4 + 1 \times 0.3 + 4 \times 0.2 = 1.2$

Variance is $E(X^2) - (E(x))^2 = 1.2 - 0.6^2 = \mathbf{0.84}$ ✓

d $\binom{5}{4} 0.1^4 \times 0.9 = \mathbf{0.00045}$✓

Use correct values for *p* and *n*✓

Question 14 [SCSA MM2019 Q18] (9 marks)

a $X \sim \text{Bin}(5, 0.05)$

states the binomial theorem✓

gives the correct parameters✓

b 1. The alarms fail independent of each other.

2. The probability that an alarm fails is constant/unchanging/same for all alarms.

states one correct assumption✓

states the second correct assumption✓

c $\text{P}(X \geq 4) = 1 - \text{P}(X \leq 3) = 1 - 0.99997 = 0.00003$

writes the probability statement correctly✓

obtains the correct final answer to at least 5 decimal places✓

d Let the random variable Y denote the number of alarms that fail out of 4.

Then $Y \sim \text{Bin}(4, 0.05)$. We need

$$\text{P}(Y \geq 3) = 0.00048$$
$$\text{P}(Y < 3) = 1 - 0.00048 = 0.99952$$

states the distribution of the random variable with correct parameters✓

writes the first probability statement correctly✓

obtains the correct final answer✓

Question 15 (5 marks)

(✓ = 1 mark)

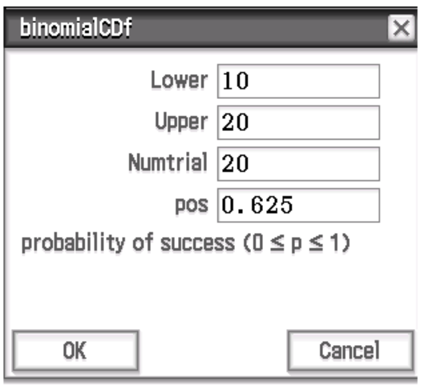
The assessment of each gym member has only two outcomes: they are assessed as fit or not fit. A member selected at random is fit with probability $p = \frac{5}{8}$ or not fit with probability $1 - p = \frac{3}{8}$.

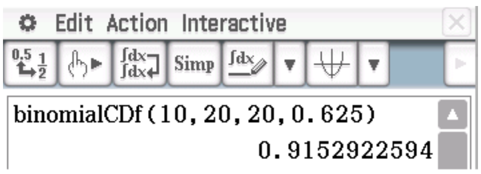
We are told that this is independent of any other member.

Hence, each assessment of a member is a Bernoulli trial. The number of members considered fit out of the 20 tested is a binomial distribution, Bin(n, p), where n = 20 and p = 0.625.

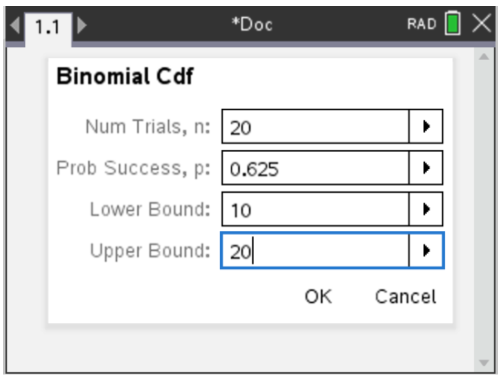
a We require P(at least 10 of 20 members are fit) = $P(X \geq 10)$.

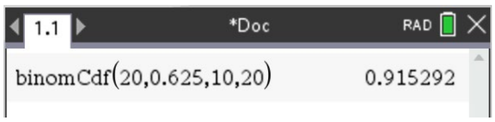
ClassPad





TI-Nspire





Using CAS, $P(X \geq 10)$ = **0.9153** ✓✓ to four decimal places.

b $$P(X > 15 \mid P(X \geq 10) = \frac{P(X > 15 \text{ and } X \geq 10)}{P(X \geq 10)} = \frac{P(X > 15)}{P(X \geq 10)}. \checkmark$$

We calculated the denominator in part **a i**.

Using CAS to obtain the numerator gives

$$P(X > 15 \mid P(X \geq 10) = \frac{0.079041409}{0.9153} \approx \mathbf{0.086}$$ ✓✓ to three decimal places.

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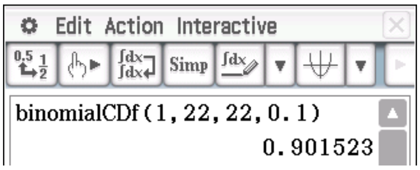
Question 16 (4 marks)

(✓ = 1 mark)

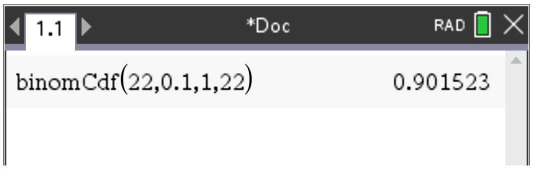
a Number of laptops L is a binomial distribution $\boldsymbol{L \sim \mathrm{Bin}(22, 0.1)}$.✓

Calculate $\mathrm{P}(L \geq 1)$ using the binomialCDf function.

ClassPad



TI-Nspire



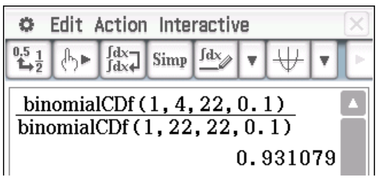
Rounding to four decimal places, the probability is **0.9015**.✓

b

$$\mathrm{P}(L<5 \mid L \geq 1)=\frac{\mathrm{P}(L<5 \cap L \geq 1)}{\mathrm{P}(L \geq 1)}$$
$$=\frac{\mathrm{P}(1 \leq L<5)}{\mathrm{P}(L \geq 1)}$$
$$=\frac{\mathrm{P}(1 \leq L \leq 4)}{\mathrm{P}(L \geq 1)}$$

Calculate $\boldsymbol{\frac{\mathrm{P}(1 \leq L \leq 4)}{\mathrm{P}(L \geq 1)}}$ using the binomialCDf function.✓

ClassPad



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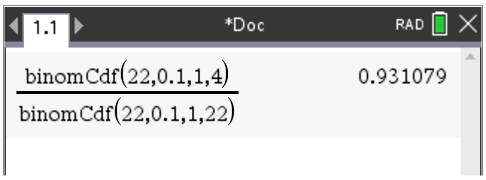

Rounding to four decimal places, the probability is **0.9311**.✓

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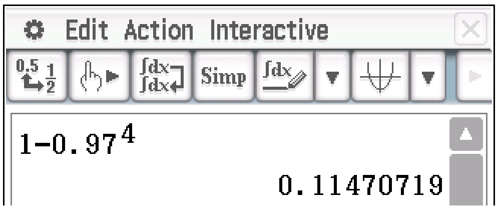
Question 17 (4 marks)

(✓ = 1 mark)

a Let X be the binomial distribution $X \sim \text{Bin}(4, 0.03)$.

Find $\text{P}(X \geq 1) = 1 - \text{P}(X = 0) = 1 - (0.97)^4$.✓

ClassPad



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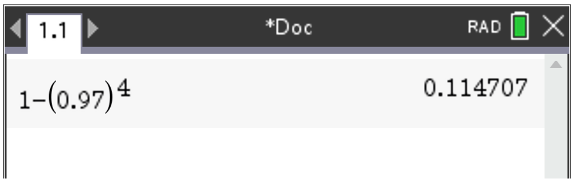
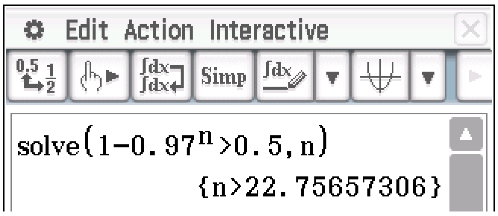


Rounding to four decimal places, the probability is **0.1147**.✓

b Solve $1 - (0.97)^n > 0.5$ for n.✓

ClassPad



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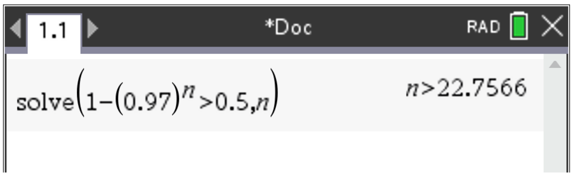

Rounding to the next whole number, the number of lemons is $\boldsymbol{n = 23}$.✓

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Question 18 [SCSA MM2016 Q20ab] (6 marks)

a

$X \sim \text{Bin}\left(30, \frac{4}{5}\right)$

$u = 24$

$s = \sqrt{30 \times \frac{4}{5}\left(1 - \frac{4}{5}\right)} = 2.19$

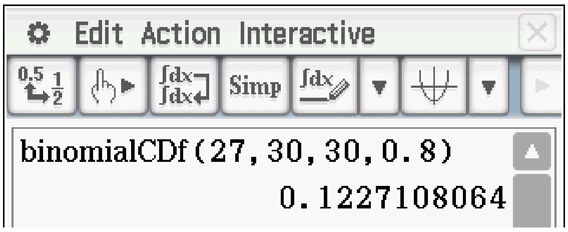
identifies binomial distribution✓

determines mean✓

determines standard deviation✓

b

ClassPad



TI-Nspire

The probability is 0.123

uses binomial distribution with correct parameters✓

determines probability✓

rounds to three decimal places✓

Question 19 (3 marks)

(✓ = 1 mark)

The assessment of each statue made by Shoddy Ltd has only two outcomes: they are assessed as Superior or Regular. We know that the probability that a statue is Regular is 0.8; hence, the probability a statue is Superior is 0.2.

Hence, each assessment of a statue is a Bernoulli trial. Let X be a random variable representing the number of Superior statues in a day's production run of n statues. X is distributed according to a binomial distribution, Bin(n, p), where $p = 0.2$.

Shoddy Ltd wants to ensure that the probability that it produces at least two Superior statues in a day's production run is at least 0.9.

We require $\mathrm{P}(X \geq 2) = 0.9$, that is, we wish to find n such that

$\mathbf{P(X = 0) + P(X = 1) \leq 0.1}$✓

$$\mathrm{P}(X=0) = {}^nC_0(0.2)^0(0.8)^n = (0.8)^n$$

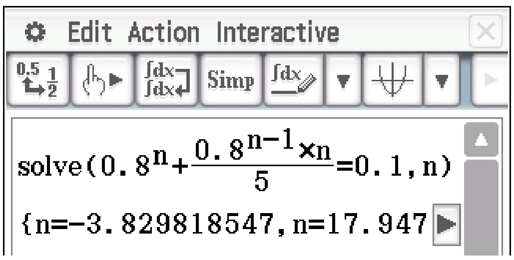
$$\mathbf{P(X=1) = {}^nC_1(0.2)^1(0.8)^{n-1} = \frac{(0.8)^{n-1}n}{5}}\checkmark$$

$$\therefore \mathrm{P}(X=0) + \mathrm{P}(X=1) = (0.8)^n + \frac{(0.8)^{n-1}n}{5} \leq 0.1$$

Use CAS to solve this equation for n. Set the calculator to Decimal (ClassPad) or Approximate (TI).

The calculators cannot solve the inequality for n, so make it an equation.

ClassPad



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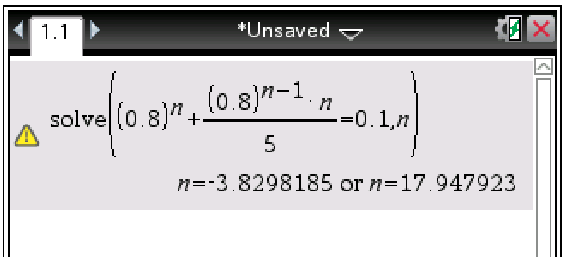

Since n is an integer greater than 17.947…, the smallest value of n, and hence the smallest number of statues, is **18 statues**.✓

Cumulative examination: Calculator-free

Question 1 (2 marks)

(✓ = 1 mark)

Use chain rule.

Let $y = f(x) = \left(x^2 + 3\right)^2$ and let $u = x^2 + 3$

Hence $\frac{dy}{du} = 2u$ and $\frac{du}{dx} = 2x$.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= 2u \times 2x$$

$$\frac{dy}{dx} = 4x^3 + 12x \checkmark$$

Hence $\mathbf{\frac{d^2y}{dx^2} = 12x^2 + 12}$ ✓

Question 2 (3 marks)

(✓ = 1 mark)

Let $P(x, 2x - 4)$ be a point on the line with equation $y = 2x - 4$.

The distance, s, from P to the origin is $s = \sqrt{x^2 + (2x-4)^2} = \sqrt{5x^2 - 16x + 16}$

For minimum s, $\frac{ds}{dx} = 0$

$$\frac{ds}{dx} = \frac{10x - 16}{2\sqrt{5x^2 - 16x + 16}} = 0 \checkmark$$

$10x - 16 = 0 \Rightarrow x = \frac{8}{5}$ and $\mathbf{y = 2 \times \frac{8}{5} - 4 = -\frac{4}{5}}$ ✓

The coordinates of P are $\mathbf{\left(\frac{8}{5}, -\frac{4}{5}\right)}$ ✓

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Question 3 [SCSA MM2020 Q1] (6 marks)

a

$$\mathrm{P}(\text{not } B) = \left(\frac{5}{8}\right)^3$$
$$= \frac{125}{512}$$

determines the correct probability✓

b

$$X \sim \mathrm{Bin}\left(3, \frac{3}{8}\right)$$

x	0	1	2	3
$\mathrm{P}(X = x)$	$\left(\frac{5}{8}\right)^3$ or $\frac{125}{512}$	$\binom{3}{1}\left(\frac{5}{8}\right)^2\left(\frac{3}{8}\right)$ or $\frac{225}{512}$	$\binom{3}{2}\left(\frac{5}{8}\right)\left(\frac{3}{8}\right)^2$ or $\frac{135}{512}$	$\left(\frac{3}{8}\right)^3$ or $\frac{27}{512}$

recognises the distribution of *X* as binomial✓

determines the correct probability for *x* = 1, 2 or 3✓

determines the correct probability for remaining entries✓

c

mean $= np$

$$= 3 \times \frac{3}{8}$$
$$= \frac{9}{8} \quad \{1.125\}$$

variance $= np(1-p)$

$$= \frac{9}{8} \times \frac{5}{8}$$
$$= \frac{45}{64}$$

determines the mean✓

determines the variance✓

Question 4 (3 marks)

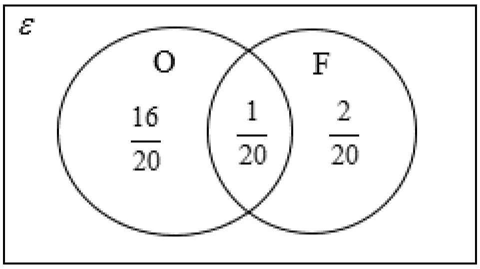
(✓ = 1 mark)

a Set up a Venn diagram

Let O represent an oil change and F represent an air filter change.

$$P(F \cap O) = \frac{1}{20}$$

$$P(O) - P(F \cap O) = \frac{17}{20} - \frac{1}{20} = \frac{16}{20}$$

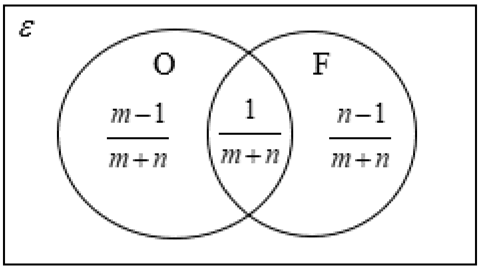


$$P(F) - P(F \cap O) = \frac{3}{20} - \frac{1}{20} = \frac{2}{20} = \frac{1}{10} = \mathbf{0.1}$$ ✓

b Converting to a fraction, $0.05=\frac{1}{20}$.

$$\mathrm{P}(F\cap O)=\frac{1}{m+n}$$

$$\mathrm{P}(O)-\mathrm{P}(F\cap O)=\frac{m}{m+n}-\frac{1}{m+n}=\frac{m-1}{m+n}$$



$$\mathrm{P}(F)-\mathrm{P}(F\cap O)=\frac{n}{m+n}-\frac{1}{m+n}=\boldsymbol{\frac{n-1}{m+n}}=\boldsymbol{\frac{1}{20}}\checkmark$$

Transpose to make m the subject.

$$\frac{n-1}{m+n}=\frac{1}{20}$$

$$m+n=20(n-1)$$

$$m+n=20n-20$$

$$\boldsymbol{m=19n-20}\checkmark$$

Question 5 (4 marks)

(✓ = 1 mark)

a

$$X \sim \text{Bin}\left(4, \frac{3}{5}\right)$$

$$\begin{aligned} P(X \geq 3) &= P(X = 3) + P(X = 4) \\ &= \mathbf{\binom{4}{3}\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right) + \left(\frac{3}{5}\right)^4} \checkmark \\ &= 4 \times \frac{27}{125} \times \frac{2}{5} + \frac{81}{625} \\ &= \frac{216}{625} + \frac{81}{625} \\ &= \mathbf{\frac{297}{625}} \checkmark \end{aligned}$$

b

$$p = \frac{3}{5},\ q = \frac{2}{5}$$

$$\begin{aligned} P(X = 2 \mid X \geq 1) &= \frac{P(X = 2) \cap P(X \geq 1)}{P(X \geq 1)} \\ &= \frac{P(X = 2)}{1 - P(X = 0)} \\ &= \frac{6 \times \left(\frac{3}{5}\right)^2 \times \left(\frac{2}{5}\right)^2}{\left(\frac{5}{5}\right)^4 - \left(\frac{2}{5}\right)^4} \\ &= \frac{\frac{6 \times 9 \times 4}{\cancel{625}}}{\frac{625 - 16}{\cancel{625}}} \\ &= \mathbf{\frac{216}{625 - 16}} \checkmark \\ &= \mathbf{\frac{6^3}{5^4 - 2^4}} \checkmark \end{aligned}$$

Examination report

Success percentage: 26%

Students were generally able to identify that conditional probability was involved. However, they need to be aware that simply quoting a rule or formula is not sufficient; they are required to demonstrate how it is used within the context of the question (i.e. in this case, give evaluations of P(X = 2) and P($X \geq$ 1). Many students did not present their answer in the required form.

Cumulative examination: Calculator-assumed

Question 1 [SCSA MM2019 Q7] (9 marks)

(✓ = 1 mark)

a $P'(t) = 2\sin(3t) + 6t\cos(3t)$ millions \$/ year

uses the product rule✓

determines correct derivative✓

b $P'(t) = 2\sin(3t) + 6t\cos(3t)$

$$P''(t) = 6\cos(3t) + 6\cos(3t) - 18t\sin(3t)$$
$$= 12\cos(3t) - 18t\sin(3t)$$

$$P''\left(\frac{\pi}{18}\right) = 12\cos\left(\frac{\pi}{6}\right) - 18t\sin\left(\frac{\pi}{6}\right)$$
$$= 6\sqrt{3} - \frac{\pi}{2} \text{ millions \$/year}^2$$

uses product rule✓

determines correct expression for the second derivative✓

substitutes into second derivative expression $\frac{\pi}{18}$ ✓

calculates the exact rate of change✓

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c $$P'\left(\frac{7\pi}{6}\right)=2\sin\left(\frac{7\pi}{2}\right)+6\left(\frac{7\pi}{6}\right)\cos\left(\frac{7\pi}{2}\right)$$
$$=-2$$

$$\delta P\approx\frac{dP}{dt}\times\delta t$$
$$\approx-2\times\frac{1}{12}$$
$$=-\frac{1}{6}$$

The approximate change in profit is $-\frac{1}{6}$ million dollars.

[$\frac{1}{6}$ million dollar loss]

calculates the correct value of P' when $t=\frac{7\pi}{6}$ ✓

states an appropriate approximation for the change in profit using the increment formula✓

substitutes and evaluates the change including units✓

Question 2 [SCSA MM2017 Q13] (9 marks)

(✓ = 1 mark)

a

Amount won	20	10	0
Probability	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{20}{36}$

completes top row correctly ✓

determines probabilities correctly✓

b Let the random variable X be the amount of money won by a player.

$$E(X) = 20 \times \frac{6}{36} + 10 \times \frac{10}{36}$$
$$= \frac{220}{36}$$
$$= \$6.11$$

writes a calculation for expected value✓

determines expected value✓

c $Expected\ payout = 6.11 - 5$

$= 1.11$

Lui Yang is better off in the long term.

In the long term Liu Yang will likely win \$1.11 per game.

determines new expected payout✓

states Lui Yang better off ✓

explains the meaning of the expected payout✓

d Let amount to be paid be $\$P$.

$$E(X) = -0.2P$$
$$-0.2P = 20 \times \frac{6}{36} + 10 \times \frac{10}{36} - P$$
$$0.8P = 6.11$$
$$P = \$7.64$$

equates $E(X)$ to $-0.2P$✓

solves to give P✓

Question 3 [SCSA MM2021 Q13abc] (6 marks)

(✓ = 1 mark)

a $X \sim \text{Bin}(5, 0.25)$

recognises the distribution is binomial✓

determines correct parameters✓

b

$$\begin{aligned} P(X=4)+P(X=5) &= \binom{5}{4}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)+\binom{5}{5}\left(\frac{1}{4}\right)^5 \\ &= \frac{15}{1024}+\frac{1}{1024} \\ &= \frac{1}{64} \end{aligned}$$

states correct probability expression✓

calculates correct probability✓

c The expected payout, E, per game is

$$E = \frac{1}{64} \times \$150 = \$2.34$$

If the carnival organisers only charge \$2 per game then on average they will lose approximately 34 c per game.

determines expected payout per game✓

concludes that charging less than the expected payout per game will lead to a loss money on average✓

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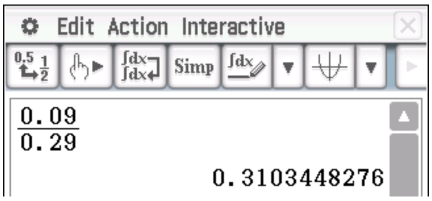
Question 4 (6 marks)

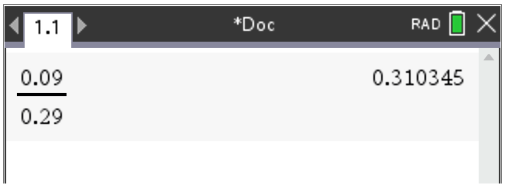
(✓ = 1 mark)

a **i** $\mathrm{P}(H \mid S) = \dfrac{\mathrm{P}(H \cap S)}{\mathrm{P}(S)} = \dfrac{0.09}{0.29}$

ClassPad



TI-Nspire

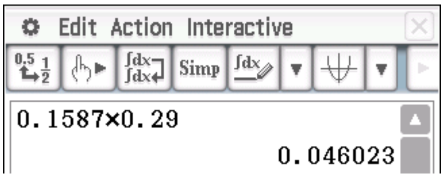


Rounding to three decimal places, the probability is **0.310**.✓

ii $\mathrm{P}(H \cap S) = 0.09$

$\mathrm{P}(H) \times \mathrm{P}(S) = 0.1587 \times 0.29 = 0.046023$

ClassPad



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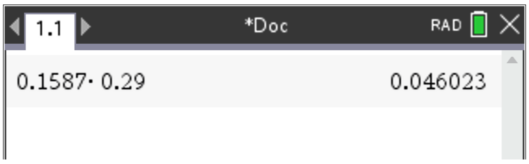
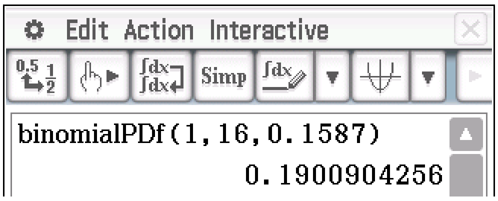


Since $0.09 \neq 0.046023$, **event *H* and event *S* are not independent**.✓

b The binomial distribution is $X \sim \mathbf{Bin}(\mathbf{16, 0.1587})$.✓

Use the binomialPDf function to calculate $\mathrm{P}(X = 1)$.

ClassPad



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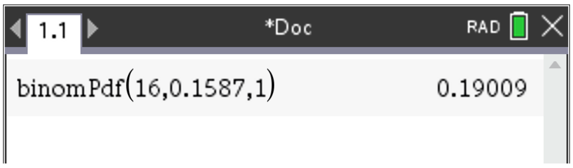

Rounding to three decimal places, the probability is **0.190**.✓

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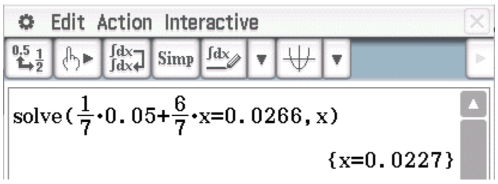
c $\frac{1}{7}$ of 5% of the students are elite so $\frac{6}{7}$ of the rest are not elite. Solve the equation $\frac{1}{7}\times 0.05+\frac{6}{7}x=0.0266$ for x.✓

ClassPad



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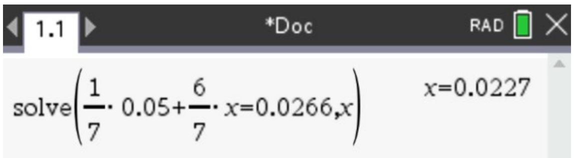

Rounding to four decimal places, the probability is **0.0227**.✓

Chapter 6 – Logarithmic functions

EXERCISE 6.1 Logarithms

Question 1

For each, write $a^x = b$ as $\log_a(b) = x$

a $\log_7(49) = 2$

b $\log_3(27) = 3$

c $\log_2(16) = 4$

d $\log_5(125) = 3$

e $\log_{11}(1) = 0$

f $\log_2(1) = 0$

g $\log_5\left(\frac{1}{25}\right) = -2$

h $\log_4\left(\frac{1}{16}\right) = -2$

Question 2

For each, write $\log_a(b) = x$ as $a^x = b$

a $5^2 = 25$

b $4^2 = 16$

c $5^3 = 125$

d $2^4 = 16$

e $3^1 = 3$

f $7^2 = 49$

g $2^7 = 128$

h $5^0 = 1$

Question 3

For each, write $\log_a(b) = x$ as $a^x = b$ and then determine x.

a $\log_2(64) = x$

$2^x = 64$

$x = 6$, since $2^6 = 64$.

$\mathbf{\log_2(64) = 6}$

b $\log_4(16) = x$

$4^x = 16$

$x = 2$, since $4^2 = 16$.

$\mathbf{\log_4(16) = 2}$

c $\log_3(81) = x$

$3^x = 81$

$x = 4$, since $3^4 = 81$.

$\mathbf{\log_3(81) = 4}$

d $\log_7(343) = x$

$7^x = 343$

$x = 3$, since $7^3 = 343$.

$\mathbf{\log_7(343) = 3}$

e $\log_6(216) = x$

$6^x = 216$

$x = 3$, since $6^3 = 216$.

$\mathbf{\log_6(216) = 3}$

f $\log_5(1) = x$

$5^x = 1$

$x = 0$, since $5^0 = 1$.

$\mathbf{\log_5(1) = 0}$

g $\log_3(3) = x$

$3^x = 3$

$x = 1$, since $3^1 = 3$.

$\mathbf{\log_3(3) = 1}$

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h $\log(100\ 000) = x$

$10^x = 100\ 000$

$x = 5$, since $10^5 = 100\ 000$.

$\mathbf{\log(100\ 000) = 5}$

i $\log_3(243) = x$

$3^x = 243$

$x = 5$, since $3^5 = 243$.

$\mathbf{\log_3(243) = 5}$

j $\log_4(1024) = x$

$4^x = 1024$

$x = 5$, since $4^5 = 1024$.

$\mathbf{\log_4(1024) = 5}$

k $\log_{\frac{1}{2}}\left(\frac{1}{16}\right) = x$

$\left(\frac{1}{2}\right)^x = \frac{1}{16}$

$\frac{1}{2^x} = \frac{1}{2^4} \Rightarrow x = 4$

$\mathbf{\log_{\frac{1}{2}}\left(\frac{1}{16}\right) = 4}$

l $\log_5\left(\frac{1}{125}\right) = x$

$5^x = \frac{1}{125} = 5^{-3}$

$x = -3$

$\mathbf{\log_5\left(\frac{1}{125}\right) = -3}$

Question 4

a $\log_4(10)+\log_4(2)-\log_4(5)$

$=\log_4\left(\dfrac{10\times 2}{5}\right)$

$=\log_4(4)$

$=\mathbf{1}$

b $\log_5(25)+\log_5(125)-\log_5(625)$

$=\log_5\left(\dfrac{25\times 125}{625}\right)$

$=\log_5(5)$

$=\mathbf{1}$

c $x=\log_8\left(\dfrac{1}{8}\right)+\log_8(4)$

$=\log_8\left(\dfrac{1}{2}\right)$

$8^x=\dfrac{1}{2}$

$(2^3)^x=2^{-1}$

$2^{3x}=2^{-1}$

$3x=-1$

$x=-\dfrac{1}{3}$

$\mathbf{\log_8\left(\dfrac{1}{8}\right)+\log_8(4)=-\dfrac{1}{3}}$

d $\log_2(16)+\log_2(4)+\log_8(8)$

$=\log_2(16\times 4\times 8)$

$=\log_2(512)$

$=\mathbf{9}$

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e $\log(400)+\log(10)-\log(4)$

$=\log\left(\frac{400\times10}{4}\right)$

$=\log(1000)$

$=\mathbf{3}$

f $\log_5(8)-\log_5(4)-\log_5(2)$

$=\log_5(8)-(\log_5(4)+\log_5(2))$

$=\log_5\left(\frac{8}{4\times2}\right)$

$=\log_5(1)$

$=\mathbf{0}$

g $\log_8(2)-\log_8\left(\frac{1}{4}\right)$

$=\log_8\left(2\div\frac{1}{4}\right)$

$=\log_8(8)$

$=\mathbf{1}$

h $\log_4(256)-\log_4(32)+\log_4(2)$

$=\log_4(256)+\log_4(2)-\log_4(32)$

$=\log_4\left(\frac{256\times2}{32}\right)$

$=\log_4(16)$

$=\log_4(4^2)$

$=\mathbf{2}$

Question 5

a $5\log_4(x)+\log_4(x^2)-\log_4(x^3)=\log_4(x^5)+\log_4(x^2)-\log_4(x^3)$

$=\log_4\left(\frac{x^5\times x^2}{x^3}\right)$

$\mathbf{=\log_4(x^4)\text{ or }4\log_4(x)}$

b $3\log_7(x)-5\log_7(x)+4\log_7(x)=\log_7(x^3)-\log_7(x^5)+\log_7(x^4)$

$=\log_7\left(\frac{x^3\times x^4}{x^5}\right)$

$\mathbf{=\log_7(x^2)\text{ or }2\log_7(x)}$

c $4\log_6(x)-\log_6(x^2)-\log_6(x^3)=\log_6(x^4)-\log_6(x^2)-\log_6(x^3)$

$=\log_6\left(\frac{x^4}{x^2\times x^3}\right)$

$\mathbf{=\log_6\left(\frac{1}{x}\right)\text{ or }-\log_6(x)}$

d $\log_2(x+2)-\log_2(x+2)^2$

$=\log_2\left(\frac{x+2}{(x+2)^2}\right)$

$\mathbf{=\log_2\left(\frac{1}{x+2}\right)}$

e $\log_4\left[(x-1)^3\right]-\log_4\left[(x-1)^2\right]$

$=\log_4\left(\frac{(x-1)^3}{(x-1)^2}\right)$

$\mathbf{=\log_4(x-1)}$

f $\log_3(x-3)+\log_3(x+3)-\log_3(x^2-9)$

$=\log_3\left(\frac{(x-3)(x+3)}{x^2-9}\right)$

$=\log_3\left(\frac{(x-3)(x+3)}{(x-3)(x+3)}\right)$

$=\log_3(1)$

$\mathbf{=0}$

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Question 6 (5 marks)

(✓ = 1 mark)

a

$$\log_2\left(\sqrt{2}\right) = \log_2\left(2^{\frac{1}{2}}\right)$$

$$= \frac{1}{2}\log_2(2)$$

$$= \frac{1}{2}\times 1$$

$$= \mathbf{\frac{1}{2}}$$ ✓

b

$$\log_9\left(9\sqrt{9}\right) = \log_9\left(9^{\frac{3}{2}}\right)$$

$$= \frac{3}{2}\log_9(9)$$

$$= \frac{3}{2}\times 1$$

$$= \mathbf{\frac{3}{2}}$$ ✓

c

$$\log_4\left(\sqrt{64}\right) = \log_4\left(4^{\frac{3}{2}}\right)$$

$$= \frac{3}{2}\log_4(4)$$

$$= \frac{3}{2}\times 1$$

$$= \mathbf{\frac{3}{2}}$$ ✓

d

$$\log_7\left(\sqrt{343}\right)=\log_7\left(7^{\frac{3}{2}}\right)$$

$$=\frac{3}{2}\log_7(7)$$

$$=\frac{3}{2}\times 1$$

$$=\mathbf{\frac{3}{2}}\checkmark$$

e

$$\log_6\left(\sqrt[3]{36}\right)=\log_6\left(6^{\frac{2}{3}}\right)$$

$$=\frac{2}{3}\log_6(6)$$

$$=\frac{2}{3}\times 1$$

$$=\mathbf{\frac{2}{3}}\checkmark$$

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EXERCISE 6.2 Exponential and logarithmic equations

Question 1

a $\log_2(32) = \log_2(2^5)$

$= 5\log_2(2)$

$= 5 \times 1$

$\mathbf{= 5}$ ✓

b $\log_5(125) = \log_5(5^3)$

$= 3\log_5(5)$

$= 3 \times 1$

$\mathbf{= 3}$ ✓

c $\log_3\left(\frac{1}{81}\right) = \log_3(3^{-4})$

$= -4\log_3(3)$

$= -4 \times 1$

$\mathbf{= -4}$ ✓

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Question 2

a $\log_2(96) - \log_2(3) = \log_2\left(\frac{96}{3}\right)$

$= \log_2(32)$

$= \log_2(2^5)$

$= 5\log_2(2)$

$= 5 \times 1$

$\mathbf{= 5}$ ✓

b $\log_5(50) + \log_5(75) - \log_5(6)$

$= \log_5\left(\frac{50 \times 75}{6}\right)$

$= \log_5(625)$

$= \log_5(5^4)$

$= 4\log_5(5)$

$= 4 \times 1$

$= 4$ ✓

Question 3

a $3^{x-5} = 7$

$x - 5 = \log_3(7)$

$\boldsymbol{x = 5 + \log_3(7)}$

b $2^{x+3} - 5 = 7$

$2^{x+3} = 12$

$x + 3 = \log_2(12)$

$\boldsymbol{x = \log_2(12) - 3}$

c $e^{3x} = 9$

$3x = \log_e(9)$

$\boldsymbol{x = \frac{1}{3}\log_e(9) = \frac{1}{3}\ln(9)}$

d $e^{2x+3} = 2$

$2x + 3 = \log_e(2)$

$\boldsymbol{x = \frac{1}{2}(\log_e(2) - 3) = \frac{1}{2}(\ln(2) - 3)}$

Question 4

a $e^x(e^x - 8) = 0$

$e^x = 0$ has no solution

$e^x - 8 = 0 \Rightarrow \boldsymbol{x = \log_e(8) = \ln(8)}$

b $5^x(5^x - 4) = 0$

$5^x = 0$ has no solution

$5^x - 4 = 0 \Rightarrow \boldsymbol{x = \log_5(4)}$

c $7(2^{3x}) - 6 = 5(2^{3x})$

$2(2^{3x}) = 6$

$2^{3x+1} = 6$

$3x + 1 = \log_2(6)$

$3x + 1 = \log_2(2 \times 3) = \log_2(2) + \log_2(3)$

$3x + 1 = 1 + \log_2(3)$

$\boldsymbol{x = \frac{1}{3}\log_2(3)}$

d $(3^x - 1)(3^{2x} - 2) = 0$

$3^x - 1 = 0 \Rightarrow \boldsymbol{x = \log_3(1) = 0}$

$3^{2x} - 2 = 0 \Rightarrow \boldsymbol{x = \frac{1}{2}\log_3(2)}$

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Question 5

a $\log_3(x+7)+\log_3(2)=3$

$\log_3(2(x+7))=3$

$2(x+7)=3^3$

$x+7=\dfrac{27}{2}$

$\boldsymbol{x=\dfrac{13}{2}}$

b $2\log_2(3)+\log_2(x+1)=4$

$\log_2(3^2(x+1))=4$

$9(x+1)=16$

$9x=7$

$\boldsymbol{x=\dfrac{7}{9}}$

c $\log_2(3x-2)+2\log_2(4)=3$

$\log_2(4^2(3x-2))=3$

$16(3x-2)=8$

$3x-2=\dfrac{1}{2}$

$\boldsymbol{x=\dfrac{5}{6}}$

d $\log_2(2x-4)+\log_2(5)=1$

$\log_2(5(2x-4))=1$

$5(2x-4)=2$

$2x-4=\dfrac{2}{5}$

$\boldsymbol{x=\dfrac{11}{5}}$

$\boldsymbol{=2\dfrac{1}{5}}$

$\boldsymbol{=2.2}$

e $\ln(x-3)-\ln(4)=0$

$\ln(x-3)=\ln(4)$

$x-3=4$

$\boldsymbol{x=7}$

f $\log_2(x+2)-\log_2(3)=3$

$\log_2\left(\frac{x+2}{3}\right)=3$

$\frac{x+2}{3}=8$

$\boldsymbol{x=22}$

Question 6

a $\log_5(x)+\log_5(3)-\log_5(2)=\log_5(6)$

$\log_5(x)+\log_5(3)-\log_5(2)-\log_5(6)=0$

$\log_5\left(\frac{x\times 3}{2\times 6}\right)=0$

$\frac{x}{4}=1$

$\boldsymbol{x=4}$

b $\log_2(x)+\log_2(6)=\log_2(3)+\log_2(x+7)$

$\log_2(6x)=\log_2(3(x+7))$

$6x=3(x+7)$

$6x=3x+21$

$\boldsymbol{x=7}$

c $\log_2(x)-3\log_2(2)=\log_2(x+1)-2\log_2(5)$

$\log_2(x)-\log_2(x+1)=3\log_2(2)-2\log_2(5)$

$\log_2\left(\frac{x}{x+1}\right)=\log_2\left(\frac{8}{25}\right)$

$\frac{x}{x+1}=\frac{8}{25}$

$25x=8x+8$

$17x=8$

$\boldsymbol{x=\frac{8}{17}}$

Question 7

a $\log_7(x)=2$

$x=7^2=49$

$\log_3(y)=4$

$y=3^4=81$

$3y-2x=3\times81-2\times49$

$\boldsymbol{=145}$

b $\log_5(x-y)-2=\log_5(2y-x)$

$\log_5(x-y)-\log_5(2y-x)=2$

$\log_5\left(\frac{x-y}{2y-x}\right)=2$

$\frac{x-y}{2y-x}=25$

$x-y=50y-25x$

$51y=26x$

$\boldsymbol{y=\frac{26}{51}x=\frac{26x}{51}}$

Question 8

a $(-1, 10)$

$a\log_3(1) + b = 10$

$0 + b = 10$

$\boldsymbol{b = 10}$

$(7, 14)$

$a\log_3(9) + 10 = 14$

$a \times 2 + 10 = 14$

$\boldsymbol{a = 2}$

b $(9, 26)$

$a\log_2(2) + b = 26$

$a + b = 26$

$(15, 36)$

$a\log_2(8) + b = 36$

$3a + b = 36$

Solve the simultaneous equations

$a + b = 26 \quad (1)$

$3a + b = 36 \quad (2)$

Subtract equation (1) from equation (2).

$3a - a = 36 - 26$

$2a = 10$

$\boldsymbol{a = 5}$

$5 + b = 26$

$\boldsymbol{b = 21}$

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Question 9 **[SCSA MM2017 Q3] (4 marks)**

(✓ = 1 mark)

$$4e^{2x} = 81 - 5e^{2x}$$
$$9e^{2x} = 81$$
$$e^{2x} = 9$$
$$\ln\left(e^{2x}\right) = \ln(9)$$
$$2x = \ln(9)$$
$$x = \frac{\ln(9)}{2}$$

collects exponential terms✓

uses natural logs to simplify the question✓

uses log laws to simplify LHS of equation✓

solves exactly for *x*✓

Question 10 **[SCSA MM2016 Q1] (5 marks)**

(✓ = 1 mark)

a $8^2 - 2^5 = 32$

determines *x* and *y*✓

recognises the inverse relationship between logarithms and exponentials✓

b
$$\log_2(x+y) + 2 = \log_2(x-2y)$$
$$\log_2(x+y) + \log_2 4 = \log_2(x-2y)$$
$$\log_2(4(x+y)) = \log_2(x-2y)$$
$$4(x+y) = x-2y$$
$$6y = -3x$$
$$y = -\frac{1}{2}x$$

expresses all terms as logarithms✓

uses log laws to combine terms✓

expresses *y* in terms of *x*✓

Question 11 **(6 marks)**

(✓ = 1 mark)

a $2\log_3(5)-\log_3(2)+\log_3(x)=2$

$\log_3\left(\frac{5^2\times x}{2}\right)=2$ ✓

$\frac{25x}{2}=9$

$x=\frac{18}{25}$ ✓

b $\log_e(3x+5)+\log_e(2)=2$

$\log_e(2(3x+5))=2$ ✓

$6x+10=e^2$

$x=\frac{1}{6}(e^2-10)=\frac{e^2-10}{6}$ ✓

c $\log_2(6-x)-\log_2(4-x)=2$

$\log_2\left(\frac{6-x}{4-x}\right)=2$ ✓

$\frac{6-x}{4-x}=4$

$6-x=16-4x$

$3x=10$

$x=\frac{10}{3}$ ✓

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Question 12 **(4 marks)**

(✓ = 1 mark)

a $f(x) = g(x)$ at (a, b)

$\mathbf{\log_3(a+1) - 3 = \log_3(2)}$ ✓

$\mathbf{\log_3(a+1) - \log_3(2) = 3}$

$\mathbf{\log_3\left(\frac{a+1}{2}\right) = 3}$ ✓

b $\log_3\left(\frac{a+1}{2}\right) = 3$

$\frac{a+1}{2} = 27$

$\mathbf{a = 53}$ ✓

The coordinates of the intersection are (a, b) and $y = \log_3(2)$

Hence $\mathbf{b = \log_3(2)}$ ✓

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EXERCISE 6.3 The logarithmic function y = $\log_a(x)$

Question 1

$\log_2(x) = 3 \Rightarrow x = 2^3 = 8$

$\log_3(y) = 2 \Rightarrow y = 3^2 = 9$

$x + y = 8 + 9 = \mathbf{17}$

Question 2

$e^{2x} + 16 = 2e^{2x}$

$e^{2x} = 16$

$2x = \log_e(16)$

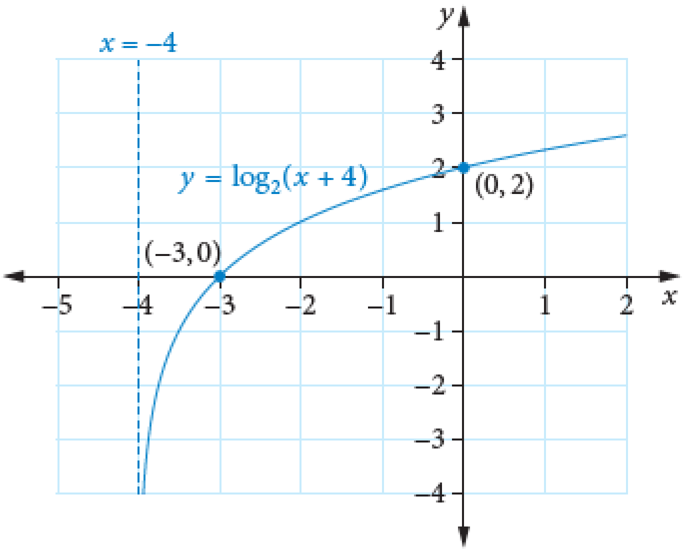
$x = \frac{1}{2}\log_e(16) = \frac{1}{2}\log_e(2^4)$

$= 2\log_e(2) = \log_e(2^2)$

$= \log_e(4) = \ln(4)$

$\boldsymbol{x = \frac{1}{2}\log_e(16)}$ or $\boldsymbol{x = 2\log_e(2)}$

Question 3



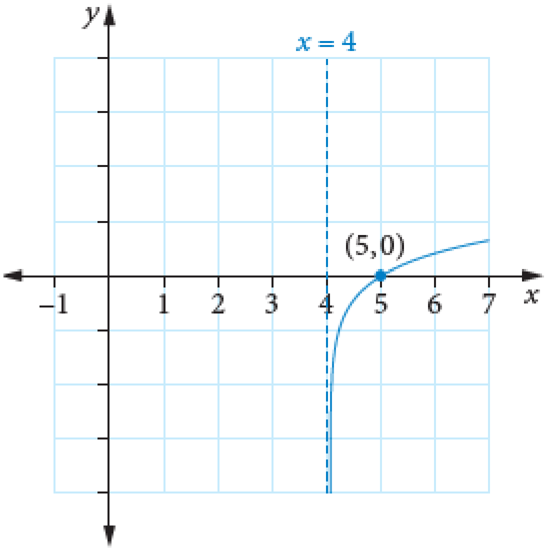
Vertical asymptote at $x = -4$ (when $x + 4 = 0$).

The x-intercept is -3 (when $x + 4 = 1$).

The y-intercept is $\log_2(0 + 4) = \log_2(4) = 2$.

The graph is a horizontal translation, 4 units left, of the graph of $f(x) = \log_2(x)$.

Question 4



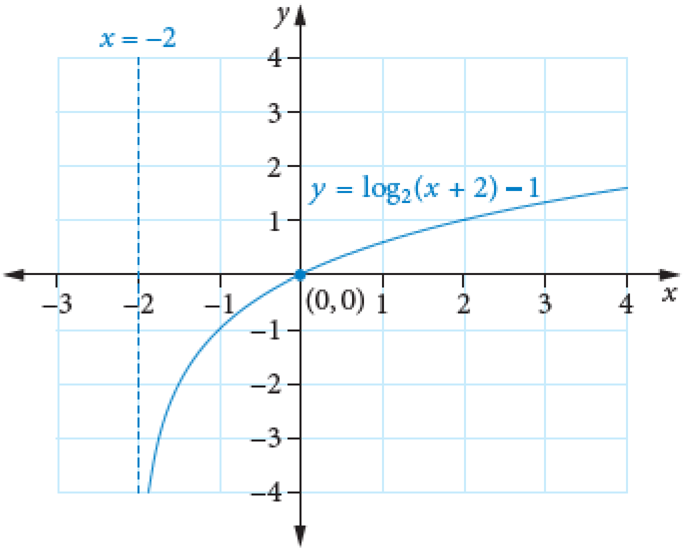
Vertical asymptote at $x = 4$ (when $x - 4 = 0$).

The x-intercept is 5 (when $x - 4 = 1$).

There is no y-intercept because the vertical asymptote is to the right of the y-axis.

The graph is a horizontal translation, 4 units right, of the graph of $f(x) = \ln(x)$.

Question 5



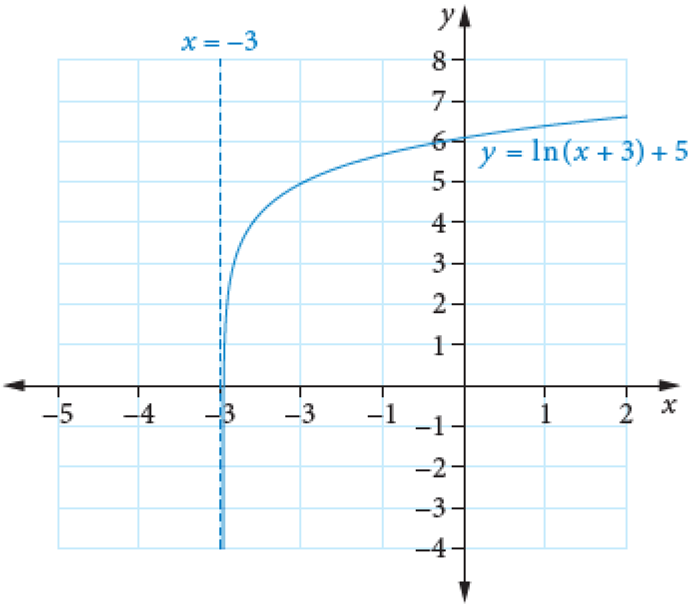
Vertical asymptote at $x = -2$ (when $x + 2 = 0$)

The x-intercept is 0 ($\log_2(x + 2) - 1 = 0$, $x + 2 = 2^1$ so $x = 0$)

The y-intercept is 0 ($\log_2(0 + 2) - 1 = 0$)

The graph is a horizontal translation (2 units left) and a vertical translation (1 unit down), of the graph of $f(x) = \log_2(x)$.

Question 6



Vertical asymptote at $x = -3$ (when $x + 3 = 0$)

The x-intercept is $e^{-5} - 3$ (when $\ln(x + 3) + 5 = 0$)

The y-intercept is $\ln(3) + 5$

The graph is a horizontal translation (3 units left) and a vertical translation (5 unit up), of the graph of $f(x) = \ln(x)$.

Question 7

$x = -b$ is the vertical asymptote. Given that $x = -4$, $-b = -4$ or $b = 4$.

The y-intercept is $a\ln(b)$ so $a\ln(4) = 6\ln(2)$.

$a\ln(4) = 2a\ln(2)$, so $2a\ln(2) = 6\ln(2)$.

Hence $a = 3$.

The rule for the function is $\mathbf{y = 3\ln(x + 4)}$.

Question 8

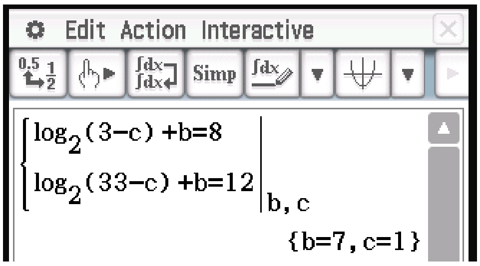
Substitute the given points to obtain a pair of simultaneous equations.

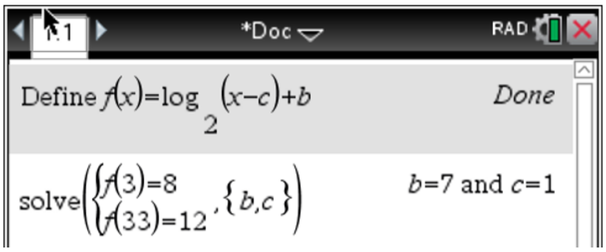
(3, 8) $\log_2(3 - c) + b = 8$

(33, 12) $\log_2(33 - c) + b = 12$

Solve the equations using a calculator.

ClassPad TI-Nspire

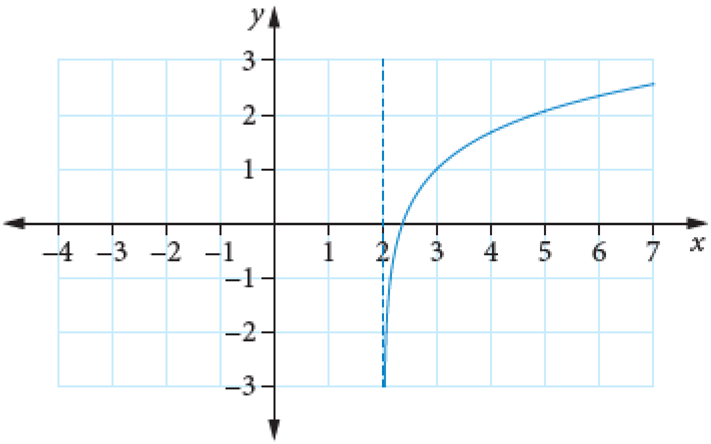




$\boldsymbol{b = 7, c = 1}$

Question 9 [SCSA MM2019 Q4b] (3 marks)

(✓ = 1 mark)



draws asymptote at $x = 2$✓

the sketch passes through the point (3, 1)✓

the sketch has the correct shape and has a y-coordiante between 2.5 and 3 when $x = 7$✓

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Question 10 (6 marks)

(✓ = 1 mark)

a Let $f(x) = 0$

$\log_3(x + 9) - 4 = 0$

$x + 9 = 3^4 = 81$

$x = 72$

The x-intercept is at (72, 0)✓

Let $x = 0$

$f(0) = \log_3(0 + 9) - 4 = -2$

The y-intercept is at (0, –2)✓

asymptote at $x + 9 = 0$ or $x = -9$✓

b Let $f(x) = 0$

$\log_2(x + 8) - 3 = 0$

$x + 8 = 2^3 = 8$

$x = 0$

The x-intercept is at (0, 0)✓

Let $x = 0$

$f(0) = \log_2(0 + 8) - 3 = 0$

The y-intercept is at (0, 0)✓

asymptote at $x + 8 = 0$ or $x = -8$✓

Question 11 (4 marks)

(✓ = 1 mark)

Vertical asymptote at $x - c = 0$ or $x = c$

Given that the vertical asymptote is at $x = 2$, then $\boldsymbol{c = 2}$.✓

Substitute (27, 10) to find b.

$\log_5(27 - 2) + b = 10$✓

$b = 10 - \log_5(25)$

$b = 10 - 2 = 8$✓

Hence $\boldsymbol{y = \log_5(x - 2) + 8}$✓

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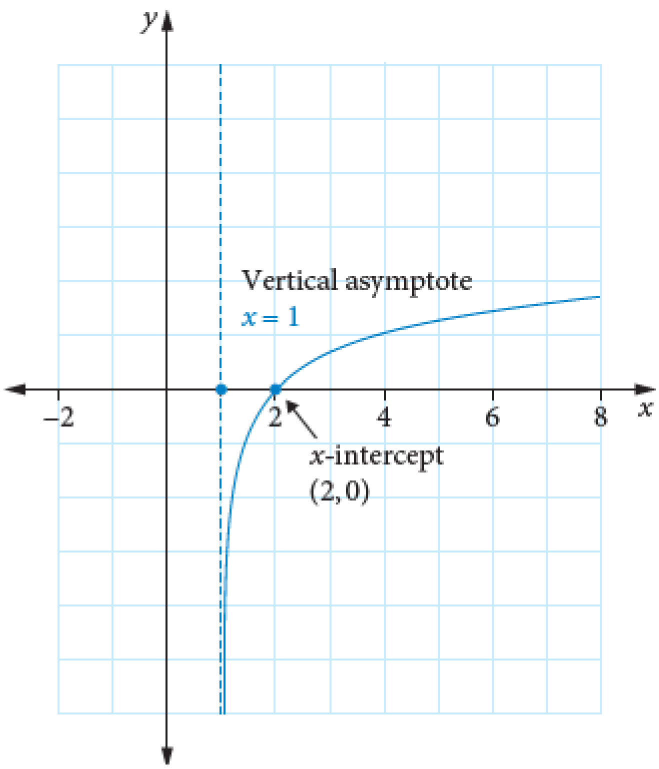
Question 12 [SCSA MM2018 Q8] (8 marks)

(✓ = 1 mark)

a



asymptote at $x = 1$✓

gives correct shape✓

x-intercept at $x = 2$✓

b $1 = \log_a(m - 1)$

$m - 1 = a$

$m = a + 1$

equates $f(m)$ to 1✓

solves for m✓

c $0 = \log_a(x - 1 + b) + c$

$-c = \log_a(x - 1 + b)$

$a^{-c} = x - 1 + b$

$x = a^{-c} + 1 - b$

coordinates are: $(a^{-c} + 1 - b, 0)$

equates new function to zero✓

solves for *x*✓

states coordinates✓

Question 13 [SCSA MM2021 Q15b] (2 marks)

(✓ = 1 mark)

Vertical asymptote at $x - p = 0$ or $x = p$

Given that the vertical asymptote is at $x = 5$, then $p = 5$.

Substituting the points into equation:

$2 = m\log_3(1) + q$ (1)

$-6 = m\log_3(9) + q$ (2)

Equation (1) gives $q = 2$

Equation (2) gives:

$-6 = m\log_3(3^2) + 2$

$-8 = 2m$

$m = -4$

substitutes the points into the equation✓

determines the values of *p*, *q* and *m*✓

EXERCISE 6.4 Applications of logarithmic functions

Question 1

$f(x)$ has an asymptote when $x - c = 0$.

Hence $10 - c = 10 \Rightarrow \boldsymbol{c = 10}$

Use (12, 21).

$$\log_2(12-10) + b = 21$$
$$\log_2(2) + b = 21$$
$$1 + b = 21$$
$$\boldsymbol{b = 20}$$

Question 2

Use (17, 4) and (11, 3) to obtain a pair of simultnaeous equations.

$$\log_3(17-c) + b = 4 \quad (1)$$

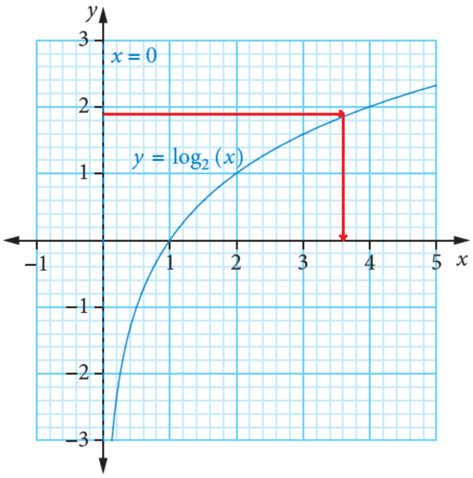
$$\log_3(11-c) + b = 3 \quad (2)$$

(1) − (2)

$$\log_3(17-c) - \log_3(11-c) = 1$$
$$\log_3\left(\frac{17-c}{11-c}\right) = 1$$
$$\frac{17-c}{11-c} = 3$$
$$17 - c = 33 - 3c$$
$$2c = 16$$
$$\boldsymbol{c = 8}$$
$$\log_3(11-8) + b = 3$$
$$1 + b = 3$$
$$\boldsymbol{b = 2}$$

Question 3

a



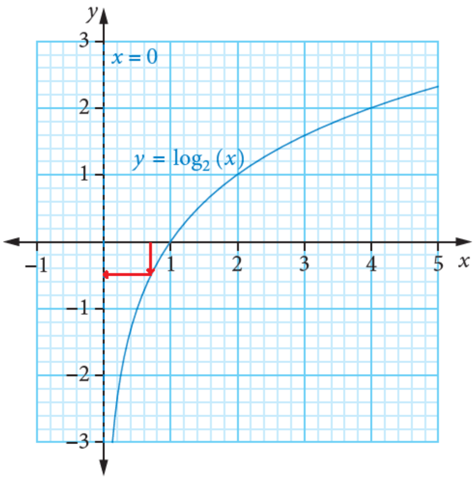
One division on the grid represent 0.2. Hence, $y = 1.9$ is $\frac{1.9}{0.2} = 9.5$ divisions.

On the x-axis, the value required is arrow points to a value equivalent to 18 squares.

This is $x = 18 \times 0.2 = 3.6$

So $\boldsymbol{n \approx 3.6}$

b



$2^{n-3} - 0.7 = 0$

$n = 3 + \log_2(0.7)$

On the x-axis, 0.7 is $\frac{0.7}{0.2} = 3.5$ divisions.

This gives –2.5 divisions on the vertical axis, which is $x = -2.5 \times 0.2 = -0.5$ on the vertical axis.

Thus $n = 3 - 0.5 = 2.5$

So $\boldsymbol{n \approx 2.5}$

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Question 4

a $C(100) = (100 - 200)\ln(0.5 \times 100 + 1) - 100 + 1000$

$= -100\ln(51) + 900$

$= 506.817...$

The monthly cost for 100 sheep is **$507**

b Solve $\frac{dC}{dx} = 0$

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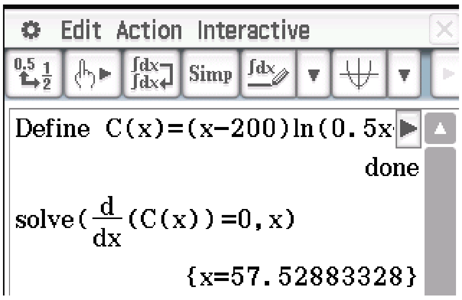

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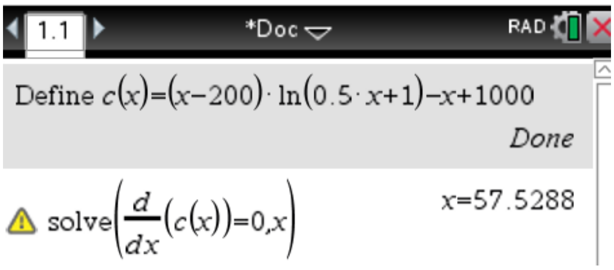
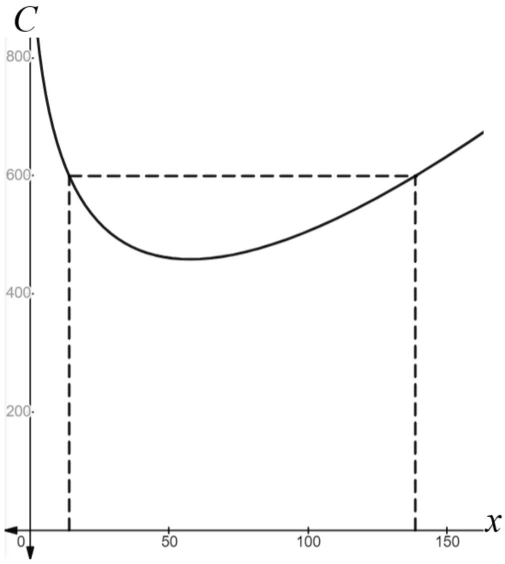


The number of sheep is 58.

c Solve $C(x) = 600$



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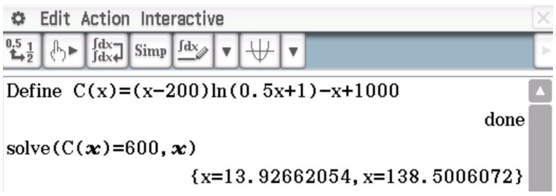

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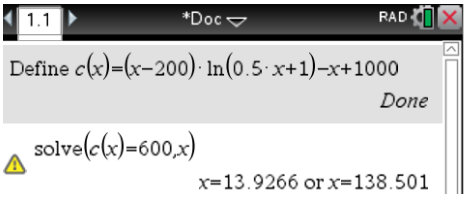

The farmer should raise 14 or 138 sheeps.

Question 5

a To find the population on 1 January, substitute $t = 0$ in the model.

$N = 500 \ln (21 \times 0 + 3)$

$= 500 \ln 3$

≈ 549.3

Hence there is a population of **549** moths on 1 January.

b To find the population after 30 days, we substitute $t = 30$ in the model.

$N = 500 \ln (21 \times 30 + 3)$

$= 500 \ln 633$

≈ 3225.24

Hence there is a population of **3225** months after 30 days.

c For finding the day when population is greater than 2000,

substitute $N = 2000$.

$2000 = 500 \ln(21t + 3)$

$\ln(21t + 3) = 4$

Taking exponents on both sides gives us

$21t + 3 = e^4$

$21t = e^4 - 3$

$$t = \frac{e^4 - 3}{21}$$

$$= 2.457\ldots$$

Hence, after **2.457 days, that is 4 January**, they have a population of more than 2000.

Question 6

$$8 = \log_{10}\left(\frac{A_1}{A_0}\right) \text{ so } \frac{A_1}{A_0} = 10^8 \text{ or } A_0 = 10^{-8} A_1$$

$$6.9 = \log_{10}\left(\frac{A_2}{A_0}\right) \text{ so } \frac{A_2}{A_0} = 10^{6.9} \text{ or } A_0 = 10^{-6.9} A_2$$

Hence $10^{-8} A_1 = 10^{-6.9} A_2$ or $A_1 = \dfrac{10^{-6.9}}{10^{-8}} A_2 \approx 12.589$

The amplitude of the earthquake in Mexico City was 12.59 larger than the amplitude of the earthquake in San Francisco Bay.

Question 7

a **i** $\log_{10}(N) = 2$

$N = 10^2 = 100$

There were 100 virus cases initially.

ii $\log_{10}(N) = 3$

$N = 10^3 = 1000$

There were 1000 virus cases after 2 days.

b B is the initial amount of cases, represented by where the graph intersects the log(N) axis. Thus, $B = 2$.

The value of A is the gradient of the straight line. Use (0, 2) and (8, 6) to find the gradient.

$$A = \frac{6-2}{8-0} = \frac{1}{2}$$

Hence, $\boldsymbol{A = \frac{1}{2}, B = 2}$

c

$$\log_{10}(N) = \frac{1}{2}t + 2$$

$t = 6,\ \log_{10}(N) = 3 + 2 = 5$

$$N = 10^5 = 100\,000$$

There were 100 000 cases after six days.

d

$$\log_{10}(N) = \frac{1}{2}t + 2$$

$$N = 10^{\frac{1}{2}t+2}$$
$$= 10^2 \times 10^{\frac{1}{2}t}$$
$$= 100 \times 10^{\frac{1}{2}t}$$

$$\boldsymbol{N = 100\left(10^{\frac{1}{2}t}\right) = 100(10)^{0.5t}}$$

Question 8 [SCSA MM2019 Q4a] (3 marks)

(✓ = 1 mark)

a $p = 4$

states the correct value of *p*✓

b $e^{p+1} = 3$

$p + 1 = \ln(3)$

$p + 1 = 1.1$

$\therefore p = 0.1$

rearranges to form a logarithmic equation✓

states the correct value of *p*✓

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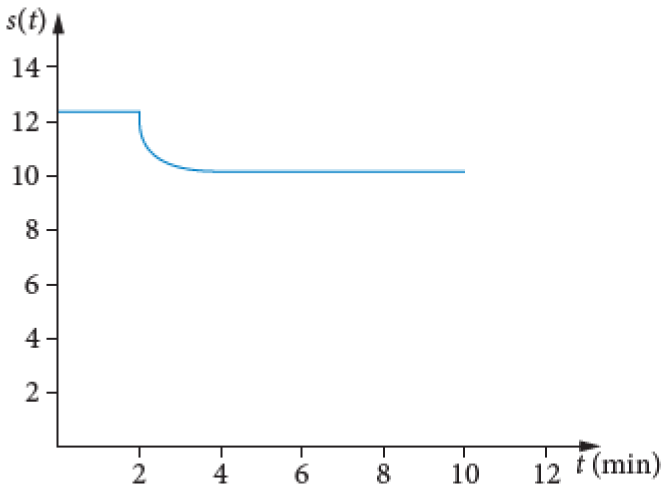
Question 9 **[SCSA MM2019 Q12] (6 marks)**

(✓ = 1 mark)

a



correctly graphs y = 12.3✓

correctly graphs $s(t)$✓

shows scale and graphs do not exceed [0, 10] domain✓

b

$$10 = 10 - \frac{\ln(t - 1.99)}{t}$$

$$0 = \frac{\ln(t - 1.99)}{t}$$

$$t = 2.99$$

Only one point in the given domain is (2.99, 10)

She runs at 10 km/h when she has run for 2.99 minutes.

states the correct time✓

c From CAS calculator: Min at (6.30, 9.77)

She has zero acceleration for the first 2 minutes of her run and at the instant t = 6.30 minutes.

states first 2 minutes✓

states 6.30 minutes✓

Question 10 **[SCSA MM2016 Q12] (3 marks)**

(✓ = 1 mark)

$$M = \log_{10} \frac{A}{A_0}$$

$$A = A_0 10^M$$

$$\frac{A_{NZ}}{A_H} = \frac{10^{5.5}}{10^{3.4}} = 10^{2.1} \approx 126$$

converts log statement to an index form✓

subtracts Richter magnitudes✓

determines ratio of amplitudes✓

Question 11 **[SCSA MM2018 Q18ab] (4 marks)**

(✓ = 1 mark)

a

$$L = 10\log\left(\frac{5\times10^9 I_0}{I_0}\right)$$

$$\approx 97 \text{ dB}$$

substitutes for *I*✓

calculates level✓

b

$$70 = 10\log\left(\frac{1\times10^{-5}}{I_0}\right)$$

$$I_0 = \frac{1\times10^{-5}}{10^7} = 1\times10^{-12} \text{ watt/m}^2$$

substitutes for *L* and *I*✓

determines I_0 including units✓

Cumulative examination: Calculator-free

Question 1 (5 marks)

(✓ = 1 mark)

$y = x + \left(x^2 - 4\right)^{\frac{1}{2}}$

$\frac{dy}{dx} = 1 + \frac{1}{2} \times 2x \times \left(x^2 - 4\right)^{-\frac{1}{2}}$

$\mathbf{= 1 + x\left(x^2 - 4\right)^{-\frac{1}{2}}}$ ✓

Use the product rule to differentiate $\frac{dy}{dx}$.

$\frac{d^2y}{dx^2} = x\left(-\frac{1}{2} \times 2x \times \left(x^2 - 4\right)^{-\frac{3}{2}}\right) + \left(x^2 - 4\right)^{-\frac{1}{2}}$

$\mathbf{= -x^2\left(x^2 - 4\right)^{-\frac{3}{2}} + \left(x^2 - 4\right)^{-\frac{1}{2}}}$ ✓

$\text{LHS} = \left(x^2 - 4\right)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y$

$\mathbf{= \left(x^2 - 4\right)\left(-x^2\left(x^2 - 4\right)^{-\frac{3}{2}} + \left(x^2 - 4\right)^{-\frac{1}{2}}\right) + x\left(1 + x\left(x^2 - 4\right)^{-\frac{1}{2}}\right) - \left(x + \left(x^2 - 4\right)^{\frac{1}{2}}\right)}$ ✓

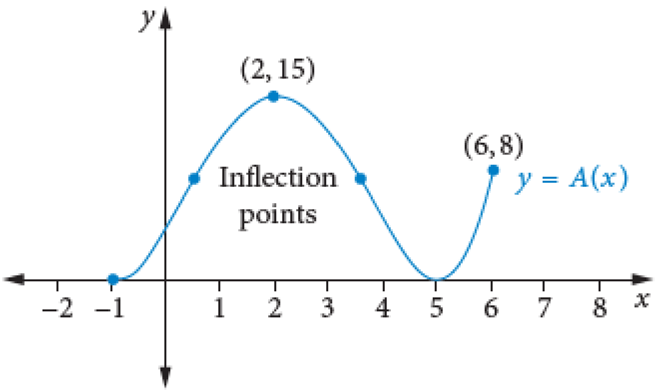
$= -x^2\left(x^2 - 4\right)^{-\frac{1}{2}} + \left(x^2 - 4\right)^{\frac{1}{2}} + x + x^2\left(x^2 - 4\right)^{-\frac{1}{2}} - x - \left(x^2 - 4\right)^{\frac{1}{2}}$ ✓

= 0✓

Hence **LHS = RHS**

Question 2 **[SCSA MM2016 Q5] (6 marks)**

(✓ = 1 mark)



sketches the function ✓

labels x-intercepts ✓✓

labels turning points ✓

labels inflection points ✓✓

Question 3 **(8 marks)**

(✓ = 1 mark)

a $e^{2x+3} = 11$

$\mathbf{2x+3=\ln(11)}$ ✓

$\mathbf{x=\frac{1}{2}(\ln(11)-3)=\frac{\ln(11)-3}{2}}$ ✓

b $5e^{2x} = 27+2e^{2x}$

$3e^{2x} = 27$

$\mathbf{e^{2x}=9}$ ✓

$2x=\ln(9)$

$\mathbf{x=\frac{1}{2}\ln(3^2)=\ln(3)}$ ✓

c $\log_2(3x-2)=4$

$\mathbf{3x-2=2^4}$ ✓

$3x=16+2$

$\mathbf{x=6}$ ✓

d $\ln(8x+4)-\ln(2)=3$

$\mathbf{\ln\left(\frac{8x+4}{2}\right)=3}$ ✓

$\frac{8x+4}{2}=e^3$

$4x+2=e^3$

$\mathbf{x=\frac{1}{4}(e^3-2)=\frac{e^3-2}{4}}$ ✓

Question 4 (3 marks)

(✓ = 1 mark)
Vertical asymptote when $x – a = 0$, so $x = a$ a is the vertical asymptote.

It is given that the vertical asymptote is at $x = –6$, so $\mathbf{a = –6}$✓.

Use (–5, 4) to find *b*.✓

$\ln(–5 – (–6)) + b = 4$

$\ln(1) + b = 4$

$\mathbf{b = 4}$✓

Cumulative examination: Calculator-assumed

Question 1 [SCSA MM2020 Q11] (9 marks)

(✓ = 1 mark)

a $f'(x) = e^x = 1$

So $x = 0, f(0) = 1$

The point of intersection is (0, 1).

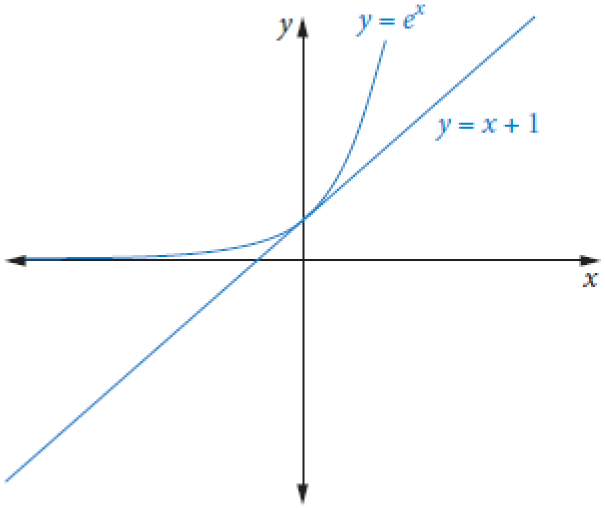
obtains equation to solve for the x coordinate✓

states the coordinates of the point✓

b The point (0, 1) lies on the line, so $1 = 0 + c \Rightarrow c = 1$

obtains the correct value of c✓

c



sketches both functions showing tangent at (0, 1) with shapes correct✓

d $$\text{Area} = \int_0^{\ln 2}\left(e^x - (x+1)\right)dx$$

$$= e^x - \frac{1}{2}x^2 - x\Big|_0^{\ln 2}$$

$$= 2 - \frac{(\ln 2)^2}{2} - \ln 2 - 1$$

$$= 1 - \frac{(\ln 2)^2}{2} - \ln 2$$

writes down the correct integrand✓

gives the correct limits for the integral✓

evaluates correctly✓

e $$\text{Area} = -\int_{\ln 2}^{0}\ln(x)\,dx$$

recognises the inverse function✓

states the correct definite integral✓

Question 2 (5 marks)

(✓ = 1 mark)

a Total slots is 37, of which 18 are black.

Use the product rule $\text{P}(\text{black}) = \frac{18}{37}$

= 0.486

b 0.486 be the probability of success in any one game.

0.514 is the probability of failure in any one game.

Require first five games to be failures and the sixth game a success.

$0.514^5 \times 0.486 = 0.017$

c This is a binomial distribution with $n = 6$, $p = 0.514$, $q = 0.486$ $x = 3$

$$P(X = x) = \binom{n}{x} p^x \times q^{n-x}$$

$$= \binom{6}{3} 0.514^3 \times 0.486^3$$

= 0.312

Question 3 **[SCSA MM2020 Q13] (7 marks)**

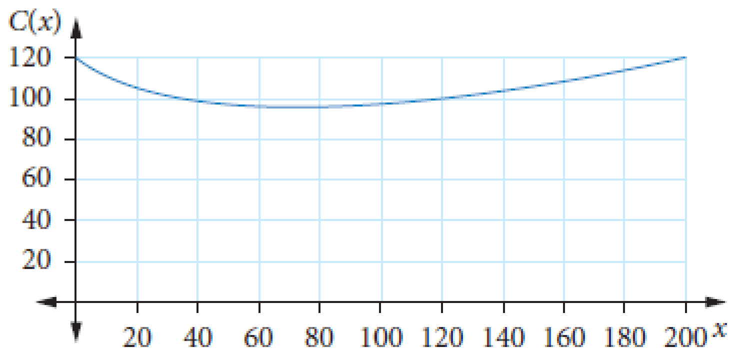
(✓ = 1 mark)

a $C(20) = \dfrac{20\ln(41)}{3} - 40 + 120 = 104.7571$

i.e. \$10 475.71 ≈ \$10 476

determines the correct cost✓

b



graph covers correct domain✓

***C*(0) and *C*(200) ae correct✓**

minimum between 70 and 80✓

c The minimum is at $x = 74.205$

$C(74) = 95.4307$ i.e. \$9543.07

$C(75) = 95.4320$ i.e. \$9543.20

The company should manufacture 74 components.

states graph is a minimum at $x = 74.205$✓

determines cost values for $x = 74$ and $x = 75$✓

states that 74 components should be manufactured per day✓

Question 4 **[SCSA MM2021 Q16 modified] (9 marks)**

(✓ = 1 mark)

a Since $P(0) = 4$, we require $\ln(a) = 1$, giving $a = e$.

recognises $P(0) = 4$✓

obtains ln(a) = 1✓

b

$$P(5)=\frac{20\ln(5+e)}{5+5}$$
$$=4.087\ (3\text{ d.p.})$$

Profit will be approximately $4 087 000

states the profit✓

c

$$P'(x)=\frac{(x+5)\dfrac{20}{(x+e)}-20\ln(x+e)}{(x+5)^2}$$

For maximum profit we require:

$$\frac{(x+5)\dfrac{20}{(x+e)}-20\ln(x+e)}{(x+5)^2}=0$$

Using CAS:

$x = 1.79$

So the maximum profit will occur in the 2nd week.

$$P(1.79)=\frac{20\ln(1.79+e)}{1.79+5}$$
$$=4.436\ (3\text{ d.p.})$$

The maximum profit will be $4 436 000.

determines when the profit was maximum✓

determines maximum profit✓

d

$$4=\frac{20\ln(x+e)}{x+5}$$
$x=0$ or 5.581

The model predicts during the 6th week.

determines when the profit falls below the 2021 value✓

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e $P(10) = 3.39072 = N(0)$

$3.39072 = 2e^{10b}$

$b = 0.05279$

$5 = 2e^{0.05279(10 + y)}$

$y \approx 7.36$

Profit should exceed \$5 million during the 8th week after the changes.

determines $P(10)$✓

determines the value of the constant b✓

determines the week when the profit exceeds \$5 million✓

Chapter 7 – Calculus of the natural logarithmic functions

EXERCISE 7.1 Differentiating logarithmic functions

Question 1

a $f(x) = 8x - 5, \quad f'(x) = 8$

$$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

$$= \mathbf{\frac{8}{8x-5}}$$

b $f(x) = 3x^2 + 6x, \quad f'(x) = 6x + 6$

$$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

$$= \mathbf{\frac{6x+6}{3x^2+6x}}$$

$$= \mathbf{\frac{2x+2}{x^2+2x}}$$

c $f(x) = \left(x^4 + 8x\right)^3, \quad f'(x) = 3\left(4x^3 + 8\right)\left(x^4 + 8x\right)^2$

$$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

$$= \mathbf{\frac{3\left(4x^3+8\right)\left(x^4+8x\right)^2}{\left(x^4+8x\right)^3}}$$

$$= \mathbf{\frac{3\left(4x^3+8\right)}{x^4+8x} = \frac{12x^3+24}{x^4+8x}}$$

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Question 2

a $f(x)=2x,\quad f'(x)=2$

$$\frac{dy}{dx}=\frac{f'(x)}{f(x)}$$

$$=\frac{2}{2x}=\frac{1}{x}$$

Or $\frac{dy}{dx}=\frac{1}{x}$ using $\frac{d}{dx}(\ln(ax))=\frac{1}{x}$

b $\frac{dy}{dx}=3\frac{d}{dx}(\ln(5x))$

$$=3\times\frac{1}{x}$$

$$=\frac{3}{x}$$

c $\frac{dy}{dx}=2\frac{d}{dx}(\ln(4x-3))$

$$=2\times\frac{4}{4x-3}$$

$$=\frac{8}{4x-3}$$

d

$f(x)=\sqrt[4]{x-4}=(x-4)^{\frac{1}{4}},\quad f'(x)=\frac{1}{4}(x-4)^{-\frac{3}{4}}$

$$\frac{dy}{dx}=\frac{f'(x)}{f(x)}$$

$$=\frac{\frac{1}{4}(x-4)^{-\frac{3}{4}}}{(x-4)^{\frac{1}{4}}}$$

$$=\frac{1}{4}(x-4)^{-1}$$

$$=\frac{1}{4(x-4)}$$

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e $f(x)=(2x+1)^3, \quad f'(x)=6(2x+1)^2$

$$\frac{dy}{dx}=\frac{f'(x)}{f(x)}$$

$$=\frac{6(2x+1)^2}{(2x+1)^3}$$

$$=\frac{6}{2x+1}$$

Question 3

a $f(x)=(x^2-2x)\ln(x)$

$$f'(x)=\frac{d}{dx}\left[x^2\ln(x)-2x\ln(x)\right]$$

$$=\frac{d}{dx}\left[x^2\ln(x)\right]-\frac{d}{dx}\left[2x\ln(x)\right]$$

$=\left[x^2\times\frac{1}{x}+2x\ln(x)\right]-\left[2x\times\frac{1}{x}+2\ln(x)\right]$ Use the product rule to differentiate terms.

$$=x+2x\ln(x)-2-2\ln(x)$$

$$=\mathbf{(2x-2)\ln(x)+x-2}$$

b $f(x)=x^3\ln(x^3)=3x^3\ln(x)$

$f'(x)=3x^3\times\frac{1}{x}+\ln(x)\times 9x^2$ Use the product rule to differentiate terms.

$$=\mathbf{3x^2+9x^2\ln(x)}$$

c $f(x)=\frac{1}{x}\ln(x)$

$f'(x)=\frac{1}{x}\times\frac{1}{x}-\frac{1}{x^2}\times\ln(x)$ Use the product rule to differentiate terms.

$=\frac{1}{x^2}-\frac{1}{x^2}\ln(x)$

$=\boldsymbol{\frac{1}{x^2}(1-\ln(x))=\frac{-\ln(x)+1}{x^2}}$

Question 4

$f(x)=\frac{\ln(2x)}{x^3}=\frac{1}{x^3}\ln(2x)$

$f'(x)=\frac{1}{x^3}\times\frac{1}{x}-\frac{3}{x^4}\times\ln(2x)$ Use the product rule to differentiate terms. The quotient rule can also be used.

$=\frac{1}{x^4}-\frac{3}{x^4}\ln(2x)$

$=\frac{1}{x^4}-\frac{3\ln(2x)}{x^4}$

$=\boldsymbol{\frac{1-3\ln(2x)}{x^4}}$

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Question 5

a $f(x)=5x+4 \Rightarrow f'(x)=5$

$$\frac{dy}{dx}=\frac{f'(x)}{f(x)}$$

$$\frac{dy}{dx}=\frac{5}{5x+4}$$

Use the quotient rule to find $\frac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2}=\frac{(5x+4)\times\frac{d}{dx}(5)-5\times\frac{d}{dx}(5x+4)}{(5x+4)^2}$$

$$=\frac{0-5\times5}{(5x+4)^2}$$

$$\boldsymbol{\frac{d^2y}{dx^2}=\frac{-25}{(5x+4)^2}}$$

b $y=2\ln\left((4x+1)^2\right)=4\ln(4x+1)$

$f(x)=4x+1 \Rightarrow f'(x)=4$

$$\frac{dy}{dx}=4\times\frac{f'(x)}{f(x)}$$

$$\frac{dy}{dx}=\frac{16}{4x+1}$$

Use the quotient rule to find $\frac{d^2y}{dx^2}$.

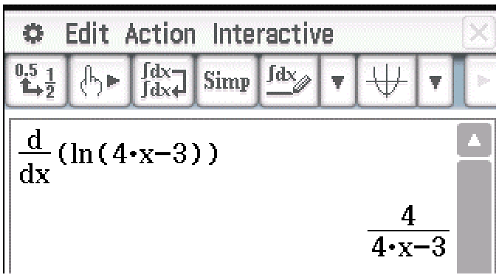
$$\frac{d^2y}{dx^2}=\frac{(4x+1)\times\frac{d}{dx}(16)-16\times\frac{d}{dx}(4x+1)}{(4x+1)^2}$$

$$=\frac{0-16\times4}{(4x+1)^2}$$

$$\boldsymbol{=\frac{-64}{(4x+1)^2}}$$

Question 6

a **ClassPad**



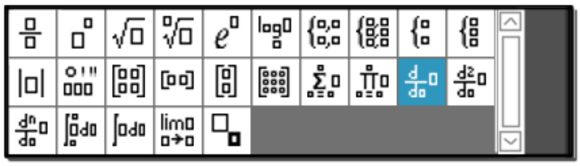
1 Enter and highlight the expression.

2 Tap Interactive >Calculus >diff.

3 Enter 1 as the order.

$$f'(x)=\frac{4}{4x-3}$$

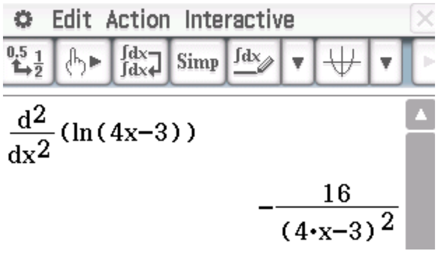
TI-Nspire



1 Press the maths template and select the derivative.

2 Enter the expression

b **ClassPad**



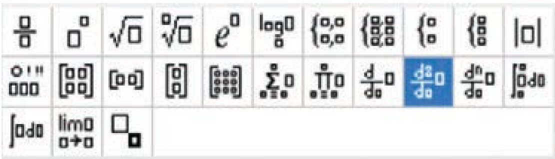
1 Enter and highlight the expression.

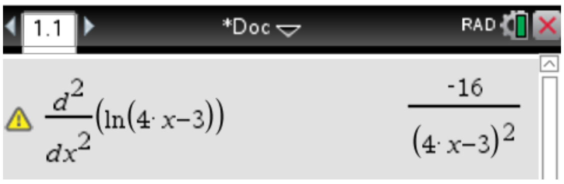
2 Tap Interactive >Calculus >diff.

3 Enter 2 as the order.

$$f''(x)=\frac{-16}{(4x-3)^2}$$

TI-Nspire





1 Press the maths template and select the second derivative.

2 Enter the expression.

Question 7 (2 marks)

(✓ = 1 mark)

Use the chain rule.

$$y = f(x) = \sin(u), \quad u = \ln(x^2) = 2\ln(x)$$

$$\frac{dy}{du} = \cos(u), \qquad \frac{du}{dx} = \frac{2}{x}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \cos(u) \times \frac{2}{x}$$ **show use of chain rule✓**

$$= \frac{2}{x}\cos\left(\ln\left(x^2\right)\right)$$

At $x = e$

$$\frac{dy}{dx} = \frac{2}{e}\cos\left(\ln\left(e^2\right)\right)$$

$$= \frac{2}{e}\cos\left(2\ln(e)\right)$$

$$= \frac{2}{e}\cos(2 \times 1)$$

$$= \frac{2}{e}\cos(2) = \frac{2\cos(2)}{e}$$ ✓

Question 8 (4 marks)

(✓ = 1 mark)

a

$$f(x) = \log_e \sqrt{\frac{3x+3}{3x-2}}$$

$$= \ln \sqrt{\frac{3x+3}{3x-2}}$$

$$= \ln\left(\frac{3x+3}{3x-2}\right)^{\frac{1}{2}} \checkmark$$

$$= \frac{1}{2}\ln\left(\frac{3x+3}{3x-2}\right) \checkmark$$

$$= \frac{1}{2}\left[\ln(3x+3) - \ln(3x-2)\right]$$

$$= \frac{1}{2}\ln(3x+3) - \frac{1}{2}\ln(3x-2) \checkmark$$

b

$$f'(x) = \frac{1}{2}\left[\frac{1}{3x+3}\times 3 - \frac{1}{3x-2}\times 3\right]$$

$$= \frac{3}{2}\left[\frac{1}{3x+3} - \frac{1}{3x-2}\right]$$

$$= \frac{3}{2}\left[\frac{\cancel{3x} - 2 - \cancel{3x} - 3}{(3x+3)(3x-2)}\right]$$

$$= \frac{\cancel{3}}{2}\left[-\frac{5}{\cancel{3}(x+1)(3x-2)}\right]$$

$$= \frac{-5}{2(x+1)(3x-2)}$$

Hence at $x = 2$, $f'(x)$ becomes

$$f'(2) = \frac{-5}{2(2+1)(3\times 2-2)}$$

$$= -\frac{5}{2\times 3\times 4}$$

$$= -\frac{5}{24} \checkmark$$

Question 9 (2 marks)

(✓ = 1 mark)

$y = x^2 \ln(x)$

To differentiate $x^2 \ln(x)$ use the product rule: identify u and v.

$u = x^2$ and $v = \ln(x)$

$\frac{du}{dx} = 2x$ and $\boldsymbol{\frac{dv}{dx} = \frac{1}{x}}$ ✓

$\frac{d}{dx}(uv) = x^2 \times \frac{1}{x} + \ln(x) \times 2x$

$\boldsymbol{\frac{dy}{dx} = x + 2x\ln(x)}$ ✓

Question 10 (2 marks)

(✓ = 1 mark)

Given $y = x\ln(x)$, differentiate using the product rule.

Identify u and v: $u = x$ and $v = \ln(x)$

Differentiate to obtain u' and v': $\boldsymbol{u' = 1}$ and $\boldsymbol{v' = \frac{1}{x}}$ ✓

Write down the expression for $uv' + vu'$ and simplify:

$\frac{dy}{dx} = uv' + vu' = x \times \frac{1}{x} + \ln(x) \times 1 = \boldsymbol{1 + \ln(x)}$ ✓

Question 11 (3 marks)

(✓ = 1 mark)

a To differentiate $\frac{\ln(x)}{x^2}$ use the product rule: identify u and v.

$u = \frac{1}{x^2}$ and $v = \ln(x)$

$\frac{du}{dx} = -2x^{-3} = \frac{-2}{x^3}$

$\frac{dv}{dx} = \frac{1}{x}$

$\frac{d}{dx}(uv) = \frac{1}{x^2} \times \frac{1}{x} + \ln(x) \times \frac{-2}{x^3}$

$f'(x) = \frac{1}{x^3} + \frac{-2\ln(x)}{x^3}$

$\boldsymbol{f'(x) = \frac{1-2\ln(x)}{x^3} = \frac{1-2\log_e(x)}{x^3}}$ ✓

b $f'(1) = \frac{1-2\ln(1)}{1^3} = \frac{1-0}{1}$

$\boldsymbol{f'(1) = 1}$ ✓

Question 12 (2 marks)

(✓ = 1 mark)

Given $f(x) = \log_e(x^2+1)$, differentiate using the chain rule.

Identify u and write f in terms of u: $u = x^2+1$, $f(u) = \log_e(u)$.

Write $\frac{du}{dx}$ and $\frac{df}{du}$ in terms of x: $\frac{du}{dx} = 2x$ and $\frac{df}{du} = \frac{1}{u} = \frac{1}{x^2+1}$.

$\frac{df}{dx} = \frac{df}{du} \times \frac{du}{dx} = \frac{1}{x^2+1} \times 2x = \frac{2x}{x^2+1}$.

Hence, the derivative of $f(x)$ is $\boldsymbol{f'(x) = \frac{2x}{x^2+1}}$.✓

This is used to calculate $f'(2) = \frac{2\times 2}{2^2+1} = \boldsymbol{\frac{4}{5}} = \mathbf{0.8}$.✓

Question 13 [SCSA MM2016 Q13a] (2 marks)

(✓ = 1 mark)

$$\begin{aligned}\frac{d}{dx}\left(x^3\ln(2x)\right) &= x^3\times\frac{1}{x}+\ln(2x)\times\left(3x^2\right)\\ &= x^2\left(1+3\ln(2x)\right)\\ &= x^2+3x^2\ln(2x)\end{aligned}$$

uses product rule✓

determines derivative✓

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EXERCISE 7.2 Applications of derivatives of the natural logarithmic function

Question 1

a

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(5-2x)}{5-2x}$$

$$= \frac{\mathbf{-2}}{\mathbf{5-2x}}$$

b

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(x^3+x^2)}{x^3+x^2}$$

$$= \frac{\mathbf{3x^2+2x}}{\mathbf{x^3+x^2}}$$

Question 2

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(x+6)}{x+6} = \frac{1}{x+6}$$

$$\frac{d^2y}{dx^2} = \frac{(x+6)\frac{d}{dx}(1) - 1\frac{d}{dx}(x+6)}{(x+6)^2}$$, quotient rule.

$$= \frac{\mathbf{-1}}{\mathbf{(x+6)^2}}$$

Question 3

a

$$f'(x)=\frac{\frac{d}{dx}\left(8x-x^2\right)}{8x-x^2}=\frac{8-2x}{8x-x^2}$$

Stationary point when $f'(x)=0$.

$$\frac{8-2x}{8x-x^2}=0\Rightarrow 8-2x=0$$
$$x=4$$

$$f(4)=\ln\left(8\times4-4^2\right)$$
$$=\ln(16)=2\ln(4)$$

Stationary point at $\mathbf{(4,2\ln(4))=(4,\ln(16))}$

b

$$f''(x)=\frac{\left(8x-x^2\right)\frac{d}{dx}(8-2x)-(8-2x)\frac{d}{dx}\left(8x-x^2\right)}{\left(8x-x^2\right)^2}$$, quotient rule

$$=\frac{-2\left(8x-x^2\right)-(8-2x)(8-2x)}{\left(8x-x^2\right)^2}$$

$$=\frac{-16x+2x^2-4x^2+32x-64}{\left(8x-x^2\right)^2}$$

$$=\frac{-2x^2+16x-64}{\left(8x-x^2\right)^2}$$

$$f''(4)=\frac{-2(4)^2+16(4)-64}{\left(8\times4-4^2\right)^2}=-\frac{1}{8}$$

Since $f''(4)<0$, $(4,2\ln(4))$ is a local maximum.

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Question 4

a

$$N'(t) = 200 \times \frac{\frac{d}{dt}\left(-t^2 + 12t + 13\right)}{-t^2 + 12t + 13} = 200 \times \frac{-2t + 12}{-t^2 + 12t + 13}$$

$$= \boldsymbol{\frac{400(6-t)}{-t^2 + 12t + 13}}$$

b Maximum when $N'(t) = 0$

$$\frac{400(6-t)}{-t^2 + 12t + 13} = 0$$

$$400(6-t) = 0 \Rightarrow t = 6$$

The population of frogs reaches its maximum in the 6th week.

c $\boldsymbol{N(6) = 200\ln\left(-6^2 + 12(6) + 13\right) \approx 778}$ **frogs.**

Question 5

$f(x) = \ln\left(x + e^2\right)$ and $f(0) = \ln\left(e^2\right) = 2\ln(e) = 2$

$f'(x) = \dfrac{1}{x + e^2}$ so $f'(0) = \dfrac{1}{e^2}$.

Hence the slope is given by $m = \dfrac{1}{e^2}$.

Put the values into the equation of the tangent line, $y = mx + c$, $y = 2$, $m = \dfrac{1}{e^2}$ and $x = 0$

we get $2 = \dfrac{1}{e^2} \times 0 + c$ so $c = 2$

Hence, the equation is $\boldsymbol{y = \dfrac{1}{e^2}x + 2}$

Question 6

$f(x)=\ln(x)$

$f'(x)=\frac{1}{x}$ so $f'(3)=\frac{1}{3}$.

Hence the slope is given by $m=\frac{1}{3}$.

Put the values into the equation of the tangent line, $y=mx+c$, $y=\ln(3)$, $m=\frac{1}{3}$ and $x=3$ we get $\ln(3)=\frac{1}{3}\times 3+c$

$c=\ln(3)-1$

Hence, the equation is $\boldsymbol{y=\frac{1}{3}x-1+\ln(3)}$

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Question 7

To find the equation of the tangent, first find $f'(x)$.

$$f(x) = 3\ln(x-2)$$

$$f'(x) = 3 \times \frac{1}{x-2}$$

$$= \frac{3}{x-2}$$

When the tangent touches the x-axis, $y = 0$.

So, we get from the equation of curve,

$$0 = 3\ln(x-2)$$

$$\ln(x-2) = 0$$

We know that $\ln(1) = 0$

So, $x - 2 = 1$

$x = 3$

So, $f'(3) = \frac{3}{3-2} = 3$

Hence the slope is given by $m = 3$.

Putting the values in the equation of the tangent line, $y = mx + c$, $y = 0$, $m = 3$ and $x = 3$ we get $0 = 3 \times 3 + c$

$c = -9$

Hence, the equation is $\boldsymbol{y = 3x - 9}$

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Question 8

$y=\ln(x)\Rightarrow\frac{dy}{dx}=\frac{1}{x}$.

$\frac{dy}{dx}\approx\frac{\delta y}{\delta x}$

$\delta y\approx\frac{dy}{dx}\delta x$

$\delta y\approx\frac{1}{3}\times 0.003=0.001$

Hence $\ln(3.003)\approx 1.0986+0.001$

$=1.0996$

Question 9

a

Speed is $x'(t)=8\frac{\frac{d}{dt}(2t+1)}{2t+1}$.

$$\mathbf{=\frac{16}{2t+1}}$$

b Solve $x'(t)=4$ to find t.

$\frac{16}{2t+1}=4$

$16=8t+4$

$=1.5$

Simon has been rowing for 1.5 hours.

c

Acceleration is $x''(t)=\frac{(2t+1)\frac{d}{dt}(16)-16\frac{d}{dt}(2t+1)}{(2t+1)^2}$

$=\frac{-32}{(2t+1)^2}$

$x''(1)=\frac{-32}{3^2}=-\frac{32}{9}$

The acceleration is $\frac{32}{9}$ km/h²

Question 10 [SCSA MM2018 Q6] (8 marks)

(✓ = 1 mark)

a The company will make a loss for a selling price between $1.50 and $2.00. The profit then increases up to approximately $2.25 per item sold for a selling price of approximately $3.25, and then decreases steadily to a value of less than $1 per item sold for a selling price of $10.

states initially making a loss✓

states profit increases to maximum at $3.25✓

states it decreases after that✓

b

$$\frac{dP}{dx} = \frac{x^2\left(\frac{50}{2}\times\frac{2}{x}\right) - 2x\times 50\ln\left(\frac{x}{2}\right)}{x^4}$$

$$= \frac{50x - 100x\ln\left(\frac{x}{2}\right)}{x^4}$$

$$= \frac{50 - 100\ln\left(\frac{x}{2}\right)}{x^3}$$

For maximum, $\frac{dP}{dx} = 0 \Rightarrow 0 = \frac{50 - 100\ln\left(\frac{x}{2}\right)}{x^3}$

$$\ln\left(\frac{x}{2}\right) = \frac{1}{2}$$

$$x = 2e^{\frac{1}{2}}$$

correctly states the numerator of the quotient rule✓

correctly states the denominator of the quotient rule✓

simplifies derivative✓

equates simplified derivative to zero✓

determines exact value of *x*✓

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Question 11 [SCSA MM2021 Q1b] (3 marks)

(✓ = 1 mark)

$$f''(x) = x \times \frac{2}{2x} + 1 \times \ln(2x)$$
$$= 1 + \ln(2x)$$

identifies the rate of change as $f''(x)$ ✓

correctly determines $f''(x)$ ✓

simplifies the expression for $f''(x)$ ✓

Question 12 [SCSA MM2021 Q3] (3 marks)

(✓ = 1 mark)

Let $y = \ln(x)$

Then $x = 2, \delta x = 0.02$

$$\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$$
$$\delta y \approx \frac{dy}{dx}\delta x$$
$$= \frac{1}{x} \times \delta x$$
$$= \frac{0.02}{2}$$
$$= 0.01$$
$$\therefore \ln(2.02) \approx \ln(2) + 0.01$$
$$\delta x = 0.703$$

correctly determines δx ✓

correctly determines δy ✓

determines correct approximation✓

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Question 13 [SCSA MM2016 Q13ab] (5 marks)

(✓ = 1 mark)

a $\frac{d}{dx}\left(x^2 \ln x\right) = x^2 \times \frac{1}{x} + \ln x \times (2x)$

$= x(1 + 2\ln x)$

uses product rule✓

determines derivative✓

b $\frac{dy}{dx} = x(1 + 2\ln x)$

$\frac{dy}{dx} = 0, \ln x = -\frac{1}{2}, \quad x \neq 0$

Only one point where derivative is zero, hence only one stationary point.

equates derivative to zero✓

states that $x \neq 0$ ✓

shows that only stationary point occurs for $\ln x = -\frac{1}{2}$ ✓

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Question 14 (9 marks)

(✓ = 1 mark)

a

$$x'(t)=\frac{d}{dt}\left(\frac{18\ln(2t+1)}{5}\right)$$

$$=\frac{18}{5}\times\frac{d}{dt}(\ln(2t+1))$$

$$=\mathbf{\frac{18}{5}\times\frac{2}{2t+1}}\ ✓$$

$$=\mathbf{\frac{36}{5(2t+1)}}\ ✓$$

b

$$x''(t)=\frac{\mathbf{5(2t+1)\frac{d}{dt}(36)-36\frac{d}{dt}(5(2t+1))}}{\mathbf{25(2t+1)^2}}$$ ✓, quotient rule

$$=\frac{-360}{25(2t+1)^2}$$

$$=\mathbf{\frac{-72}{5(2t+1)^2}}\ ✓$$

c

$$x''(2)=\frac{-72}{5(5)^2}=-\mathbf{\frac{72}{125}}$$

The runner's accelaration after 2 hours is $-\frac{72}{125}$ km/h^2 or −0.576 km/h^2.✓

d **Use a calculator to solve the equation $x''(t)=-1$ to find *t*.**✓

$$\mathbf{\frac{-72}{5(2t+1)^2}=-1}\ ✓$$

Take the positive solution, $t\approx 1.3974$ hours or ***t* = 1 hour 24 minutes.**✓

EXERCISE 7.3 Integrals producing natural logarithmic functions

Question 1

$$f(x) = e^x \ln(x)$$

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(e^x \ln(x))$$

$$= e^x \times \frac{1}{x} + e^x \ln(x)$$

$$= e^x \left[\frac{1 + x\ln(x)}{x}\right]$$

$$= \frac{e^x + xe^x \ln(x)}{x} \text{ or } \frac{e^x}{x} + e^x \ln(x)$$

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Question 2

$f(x)=\log_e(x^3+1)$

So, $f'(x)=\frac{d}{dx}\left(\log_e(x^3+1)\right)$

$=\frac{1}{x^3+1}\times 3x^2$

$=\frac{3x^2}{x^3+1}$

For finding $f'(2)$, substitute the value of $x=2$.

$f'(x)=\frac{3x^2}{x^3+1}$

$f'(2)=\frac{3\times 2^2}{2^3+1}$

$=\frac{12}{8+1}$

$=\frac{12}{9}$

$=\mathbf{\frac{4}{3}}$

Question 3

a $\int\frac{2}{x}dx=2\int\frac{1}{x}dx$

Use $\int\frac{1}{ax+b}dx=\frac{1}{a}\log_e(ax+b)+c$

$\int\frac{2}{x}dx=2\int\frac{1}{x}dx$

$=\mathbf{2\log_e(x)+c=2\ln(x)+c}$

b $\int\frac{6}{5x}dx=\frac{6}{5}\int\frac{1}{x}dx$

$=\mathbf{\frac{6}{5}\log_e(x)+c=\frac{6}{5}\ln(x)+c}$

c $\int \frac{dx}{3x} = \frac{1}{3}\int \frac{1}{x}dx$

$$= \mathbf{\frac{1}{3}\log_e(x) + c = \frac{1}{3}\ln(x) + c}$$

Question 4

a Let $f(x) = x^2 + 11x - 15$, so $f'(x) = 2x + 11$

$$\int \frac{2x+11}{x^2+11x-15}dx = \int \frac{f'(x)}{f(x)}dx$$

$$= \ln(f(x))$$

$$= \mathbf{\ln(x^2 + 11x - 15) + c}$$

b Let $f(x) = x^3 - 13$, so $f'(x) = 3x^2$

$$\int \frac{15x^2}{x^3-13}dx = 5\int \frac{3x^2}{x^3-13}dx$$

$$= 5\int \frac{f'(x)}{f(x)}dx$$

$$= 5\ln(f(x))$$

$$= \mathbf{5\ln(x^3 - 13) + c}$$

c Let $f(x) = 3x^3 + 4x^2 + 1$, so $f'(x) = 9x^2 + 8x$

$$\int \frac{18x^2+16x}{3x^3+4x^2+1}dx = 2\int \frac{9x^2+8x}{3x^3+4x^2+1}dx$$

$$= 2\int \frac{f'(x)}{f(x)}dx$$

$$= 2\ln(f(x))$$

$$= \mathbf{2\ln(3x^3 + 4x^2 + 1) + c}$$

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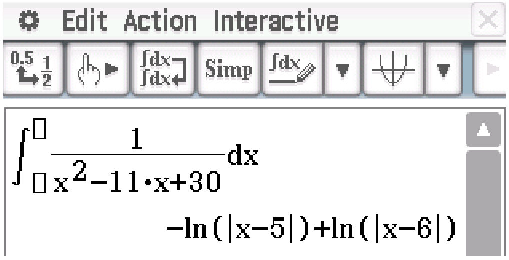
Question 5

a ClassPad



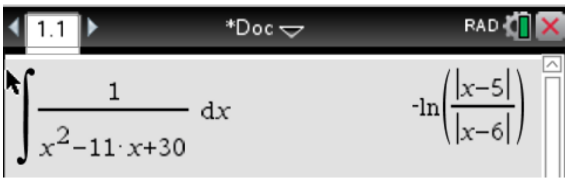
1 Enter and highlight the expression.

2 Tap Interactive > Calculus > ∫.

3 Tap OK.

$$\int \frac{1}{x^2 - 11x + 30}\,dx = \ln(|x-6|) - \ln(|x-5|) + c$$

TI-Nspire



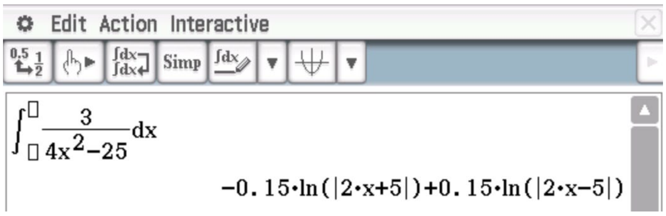
Press menu > calculus > integral.

2 Enter the expression, including the dx.

$$\int \frac{1}{x^2 - 11x + 30}\,dx = -\ln\left(\frac{|x-5)|}{|x-6|}\right) + c$$

Note that both answers are correct. The answer using TI-Nspire can be obtained by combining the logs in the solution given by ClassPad.

b ClassPad



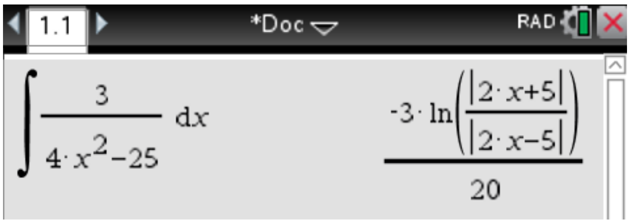
1 Enter and highlight the expression.

2 Tap Interactive > Calculus > ∫.

3 Tap OK.

$$\int \frac{3}{4x^2 - 25}\,dx = 0.15\ln(|2x-5|) - 0.15\ln(|2x+5|) + c$$

TI-Nspire



Press menu > calculus > integral.

2 Enter the expression, including the dx.

$$\int \frac{3}{4x^2 - 25}\,dx = -0.15\ln\left(\frac{|2x+5)|}{|2x-5|}\right) + c$$

Question 6

a Use the product rule.

$$f'(x) = x^2 \frac{d}{dx}\left(\log_e(2x)\right) + \frac{d}{dx}\left(x^2\right)\log_e(2x)$$

$$= x^2 \times \frac{2}{2x} + 2x\log_e(2x)$$

$$\mathbf{= x + 2x\log_e(2x)} \text{ or } \mathbf{x\left(1 + 2\log_e(2x)\right)} \text{ or } \mathbf{2x\ln(2x) + x}$$

b $$\frac{d}{dx}\left(x^2 \log_e(2x)\right) = x + 2x\log_e(2x)$$

$$\int \frac{d}{dx}\left(x^2 \log_e(2x)\right) dx = \int \left(x + 2x\log_e(2x)\right) dx$$

$$x^2 \log_e(2x) = \frac{1}{2}x^2 + 2\int x\log_e(2x)\, dx$$

$$\boldsymbol{\int x\log_e(2x)\, dx = \frac{1}{2}x^2\left(\log_e(2x) - \frac{1}{2}\right) + c = \frac{1}{2}x^2\ln(2x) - \frac{1}{4}x^2 + c}$$

Question 7

a Use the product rule.

$$f'(x) = x\frac{d}{dx}\left(\log_e\left(x^3\right)\right) + \frac{d}{dx}(x)\log_e\left(x^3\right)$$

$$= x \times \frac{3}{x} + 1 \times \log_e\left(x^3\right)$$

$$\mathbf{= 3 + \log_e\left(x^3\right)}$$

b $$\frac{d}{dx}\left(x\log_e\left(x^3\right)\right) = 3 + \log_e\left(x^3\right)$$

$$\int \frac{d}{dx}\left(x\log_e\left(x^3\right)\right) dx = \int \left(3 + \log_e\left(x^3\right)\right) dx$$

$$x\log_e\left(x^3\right) = 3x + \int \log_e\left(x^3\right) dx$$

$$\boldsymbol{\int \log_e\left(x^3\right) dx = x\log_e\left(x^3\right) - 3x + c}$$

Question 8

a $\int \frac{1}{5x+3}\,dx = \frac{1}{5}\int \frac{5}{5x+3}\,dx$

$= \frac{1}{5}\int \frac{f'(x)}{f(x)}\,dx$, where $f(x) = 5x+3$

$\mathbf{= \frac{1}{5}\ln(5x+3)+c}$

b $\int \frac{3}{2x-5}\,dx = \frac{3}{2}\int \frac{2}{2x-5}\,dx$

$= \frac{3}{2}\int \frac{f'(x)}{f(x)}\,dx$, where $f(x) = 2x-5$

$\mathbf{= \frac{3}{2}\ln(2x-5)+c}$

Question 9

a Find the integral.

$\int_1^5 \frac{dx}{x} = \left[\log_e(x)\right]_1^5$

$= \log_e(5) - \log_e(1)$

$= \log_e(5) - 0$

$= \log_e(5)$

$= \mathbf{\ln(5)}$

b Find the integral and substitute the limits.

$\int_2^9 \frac{dx}{x-1} = \left[\ln(x-1)\right]_2^9$

$= \ln(9-1) - \ln(2-1)$

$= \ln(8) - \ln(1)$

$= \ln(8)$

$= \ln\left(2^3\right)$

$= \mathbf{3\ln(2)}$

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c Find the integral and apply the limits.

$$\int_6^7 \frac{dx}{3x-2} = \frac{1}{3}\Big[\ln(3x-2)\Big]_6^7$$

$$= \frac{1}{3}\Big[\ln(3\times7-2) - \ln(3\times6-2)\Big]$$

$$= \frac{1}{3}\Big[\ln(19) - \ln(16)\Big]$$

$$= \mathbf{\frac{1}{3}\ln\left(\frac{19}{16}\right)}$$

d Find the integral and apply the limits.

$$\int_2^4 \frac{dx}{20-3x} = -\frac{1}{3}\Big[\ln(20-3x)\Big]_2^4$$

$$= -\frac{1}{3}\Big[\ln(20-12) - \ln(20-6)\Big]$$

$$= -\frac{1}{3}\Big[\ln(8) - \ln(14)\Big]$$

$$= \frac{1}{3}\Big[\ln(14) - \ln(8)\Big]$$

$$= \frac{1}{3}\left[\ln\left(\frac{14}{8}\right)\right]$$

$$= \mathbf{\frac{1}{3}\ln\left(\frac{7}{4}\right)}$$

e Find the integral and apply the limits.

$$\int_e^{4e} \frac{dx}{x} = \Big[\ln(x)\Big]_e^{4e}$$

$$= \ln(4e) - \ln(e)$$

$$= \ln(4) + \ln(e) - \ln(e)$$

$$= \ln(4)$$

$$= \ln(2^2)$$

$$= \mathbf{2\ln 2}$$

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Question 10

$$f(x)=\int \frac{7}{3x-5}\,dx$$
$$=\frac{7}{3}\ln(3x-5)+c$$

$f(2)=7$, so $7=\frac{7}{3}\ln(1)+c \Rightarrow c=7$

$$\boldsymbol{f(x)=\frac{7}{3}\ln(3x-5)+7}$$

Question 11

$$f(x)=\int\left(\frac{9}{x-3}+4\right)dx$$
$$=9\int \frac{1}{x-3}\,dx+\int 4dx$$
$$=9\ln(x-3)+4x+c$$

$f(4)=5$, so $5=9\ln(1)+16+c \Rightarrow c=-11$

$$\boldsymbol{f(x)=9\ln(x-3)+4x-11}$$

Question 12

$$\int_2^m \frac{3}{3x-1}dx = 7$$

$$\left[\ln(3x-1)\right]_2^m = 7$$

$$\ln(3m-1) - \ln(5) = 7$$

$$\ln\left(\frac{3m-1}{5}\right) = 7$$

Taking exponents on both sides,

$$\frac{3m-1}{5} = e^7$$

$$3m - 1 = 5e^7$$

$$3m = 5e^7 + 1$$

$$\boldsymbol{m = \frac{5e^7+1}{3}}$$

Question 13

$$\int_k^4 \frac{-1}{5-x}dx = \ln(2)$$

$$\left[\ln(5-x)\right]_k^4 = \ln 2$$

$$\left[\ln(5-4) - \ln(5-k)\right] = \ln 2$$

$$\left[\ln(1) - \ln(5-k)\right] = \ln 2$$

$$\ln\left(\frac{1}{5-k}\right) = \ln 2$$

$$\frac{1}{5-k} = 2$$

$$10 - 2k = 1$$

$$10 = 2k + 1$$

$$2k = 9$$

$$\boldsymbol{k = \frac{9}{2}}$$

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Question 14 (4 marks)

(✓ = 1 mark)

a $y = x\log_e(3x)$

Use the product rule $\boldsymbol{\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}}$ ✓

Let $u = x$ $\quad \frac{du}{dx} = 1$

Let $v = \log_e(3x)$ $\quad \frac{dv}{dx} = \frac{1}{3x} \times 3 = \frac{1}{x}$

$\frac{dy}{dx} = 1 \times \log_e(3x) + x \times \frac{1}{x}$

$\quad \boldsymbol{= \log_e(3x) + 1}$ ✓

b From part **a**, $\frac{d}{dx}(x\log_e(3x)) = \log_e(3x) + 1$ so $\int(\log_e(3x)+1)\,dx = x\log_e(3x) + c$

$\boldsymbol{\int_1^2(\log_e(3x)+1)\,dx = [x\log_e(3x)]_1^2}$ ✓

$=2\times\log_e(3\times 2) - 1\times\log_e(3\times 1)$

$=2\log_e(6) - \log_e(3)$

$=\log_e\left(\frac{36}{3}\right)$

$\boldsymbol{= \log_e(12)}$ ✓

Question 15 (5 marks)

(✓ = 1 mark)

a $\dfrac{2}{2x-1}=\dfrac{2}{2\left(x-\frac{1}{2}\right)}=\dfrac{1}{x-\frac{1}{2}}$

$$\int_4^5 \frac{1}{x-\frac{1}{2}}\,dx=\left[\mathbf{log}_e\left(\boldsymbol{x}-\frac{\mathbf{1}}{\mathbf{2}}\right)\right]_4^5 \checkmark$$

$$=\log_e\left(\frac{9}{2}\right)-\log_e\left(\frac{7}{2}\right)$$

$$=\log_e\left(\frac{9}{7}\right)$$

So, $\boldsymbol{b}=\dfrac{\mathbf{9}}{\mathbf{7}}$ ✓

b $$\int_2^3 \frac{1}{1-x}\,dx=-\int_2^3 \frac{\mathbf{1}}{\boldsymbol{x}-\mathbf{1}}\,\mathbf{d}\boldsymbol{x} \checkmark$$

$$=-\left[\mathbf{log}_e\left(\boldsymbol{x}-\mathbf{1}\right)\right]_2^3 \checkmark$$

$$=-\left(\log_e(2)-\log_e(1)\right)$$

$$=-\log_e(2)$$

$$=\log_e\left(\frac{1}{2}\right)$$

So, $\boldsymbol{p}=\dfrac{\mathbf{1}}{\mathbf{2}}$ ✓

Question 16 [SCSA MM2018 Q7] (6 marks)

(✓ = 1 mark)

a
$$\frac{d}{dx}(x\ln(x)) = x \times \frac{1}{x} + \ln(x)$$
$$= 1 + \ln(x)$$

uses product rule to determine derivative✓

simplifies the derivative✓

b
$$\frac{d}{dx}(x\ln(x)) = 1 + \ln(x)$$
$$\int \frac{d}{dx}(x\ln(x))\,dx = x + \int \ln(x)\,dx + c$$
$$\int \ln(x)\,dx = x\ln(x) - x + c$$

integrates both sides of answer from part a✓

partly integrates right-hand side to get *x*✓

uses Fundamental Theorem of Calculus to simplify the left-hand side✓

rearranges to give the required result✓

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Question 17 (7 marks)

(✓ = 1 mark)

a

$$a+\frac{b}{x-1}=\frac{ax+(b-a)}{x-1}$$

Given $\frac{x+5}{x-1}=\frac{ax+(b-a)}{x-1}$, $\boldsymbol{x+5=ax+(b-a)}$ ✓

Equate coefficients of powers of x.

$$\frac{x+5}{x-1}=\frac{ax+(b-a)}{x-1}$$

$x=ax \Rightarrow a=1$

$5=b-a \Rightarrow 5=b-1$

$b=6$

b

$$f(x)=\int\left(1+\frac{6}{x-1}\right)dx$$

$$=x+6\ln(x-1)+c$$

Use $f(2)=1$ to find c.✓

$1=2+6\ln(2-1)+c \Rightarrow \mathbf{c=-1}$ ✓

$\boldsymbol{f(x)=6\ln(x-1)+x-1}$ ✓

c

$$f'(x)=\frac{x+5}{x-1}$$

$f'(2)=7$

The gradient at $x=2$ is 7.✓

d The equation of the tangent line is $y=mx+c$.

$m=7,\ x=2,\ y=1$

$\mathbf{1=7(2)+c \Rightarrow c=-13}$ ✓

The equation of the tangent line is gradient at $y=7x-13$✓

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EXERCISE 7.4 Applications involving natural logarithms

Question 1

$$\int_0^m \frac{4}{4x+1}\,dx = \left[\ln(4x+1)\right]_0^m$$
$$= \ln(4m+1) - \ln(1)$$
$$= \ln(4m+1)$$

$\ln(4m+1) = \ln(13)$

$4m+1 = 13$

$\boldsymbol{m = 3}$

Question 2

$f(x) = \int \frac{1}{x+3}\,dx = \ln(x+3) + c$

$f(-2) = 12$

$\ln(-2+3) + c = 12$

$c = 12$

$\boldsymbol{f(x) = \ln(x+3) + 12}$

Question 3

$$\int_4^5 \frac{1}{3x-9}\,dx = \frac{1}{3}\int_4^5 \frac{3}{3x-9}\,dx$$
$$= \frac{1}{3}\Big[\ln(3x-9)\Big]_4^5$$
$$= \frac{1}{3}\big(\ln(6)-\ln(3)\big)$$
$$= \frac{1}{3}\ln(2)$$

Area is $\frac{1}{3}\ln(2)$ units^2

Question 4

$$\int_1^{e^3} \frac{4}{x}\,dx = 4\Big[\ln(x)\Big]_1^{e^3}$$
$$= 4\big(\ln(e^3)-\ln(1)\big)$$
$$= 4\big(3\ln(e)-\ln(1)\big)$$
$$= 4(3-0)$$
$$= 12$$

Area is 12 units^2

Question 5

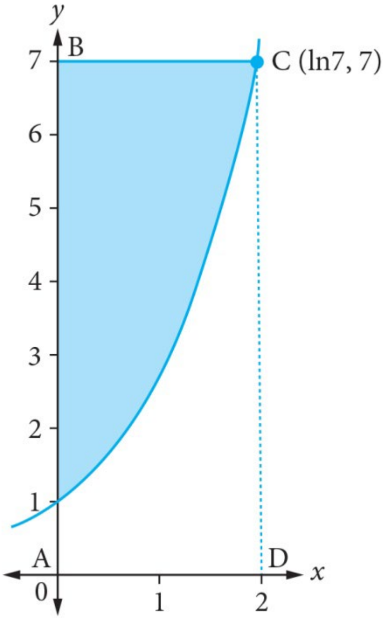
a Solve for $f(x) = g(x)$ to find x.

$e^x = 7$

$x = \ln(7)$

Hence $\boldsymbol{b = \ln(7)}$

b



Area is area of rectangle ABCD minus area below curve in (0, ln(7))

$7 \times \ln(7) - \int_0^{\ln(7)} e^x dx$

$= 7\ln(7) - \left[e^x\right]_0^{\ln(7)}$

$= 7\ln(7) - (7 - 1)$

$= 7\ln(7) - 6$

Area is $\boldsymbol{7\ln(7) - 6}$ units²

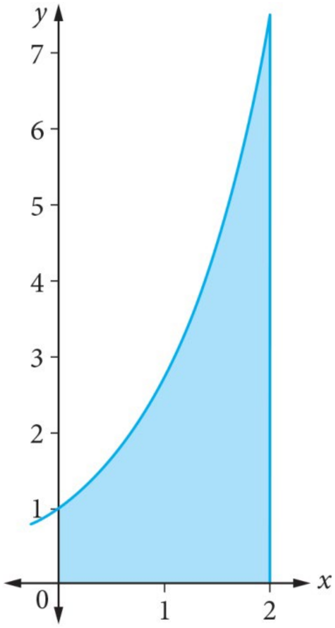
Question 6

a Solve for $f(x) = g(x)$ to find x.

$e^x = e^2 \Rightarrow x = 2$

Hence $\boldsymbol{a = 2}$

b



Area is

$\int_0^2 e^x dx$

$= \left[e^x\right]_0^2$

$= e^2 - 1$

Area is $\boldsymbol{e^2 - 1}$ units²

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Question 7 (4 marks)

(✓ = 1 mark)

a $y = x^2 \log_e(x)$

Use product rule.

$$\frac{dy}{dx} = x^2 \times \frac{d}{dx}(\log_e(x)) + \frac{d}{dx}(x^2) \times \log_e(x)$$

$$= x^2 \times \frac{1}{x} + 2x \times \log_e(x)$$

$$= \boldsymbol{x + 2x\log_e(x)}$$ ✓

b $\frac{d}{dx}(x^2 \log_e(x)) = x + 2x\log_e(x)$

$$\boldsymbol{x^2\log_e(x) = \int (x + 2x\log_e(x))\,dx}$$ ✓

$$= \frac{1}{2}x^2 + \int 2x\log_e(x)\,dx$$

$$\boldsymbol{\int_1^3 x\log_e(x)\,dx = \frac{1}{2}\left[x^2\log_e(x) - \frac{1}{2}x^2\right]_1^3 = \frac{1}{2}\left[x^2\left(\log_e(x) - \frac{1}{2}\right)\right]_1^3}$$ ✓

$$= \frac{1}{2}\left(9\left(\log_e(3) - \frac{1}{2}\right) - \times 1\left(0 - \frac{1}{2}\right)\right)$$

$$= \frac{1}{2}\left(9\log_e(3) - \frac{9}{2} + \frac{1}{2}\right)$$

$$= \frac{1}{2}(9\log_e(3) - 4)$$

$$\boldsymbol{= \frac{9}{2}\log_e(3) - 2 = \frac{9}{2}\ln(3) - 2 \text{ units}^2}$$ ✓

Question 8 [SCSA MM2017 Q5] (8 marks)

(✓ = 1 mark)

a **i** $2 = e^x$

$x = \ln(2)$

Point A has coordinates (ln(2), 2)

determines correct coordinates✓

ii The required area is the area of the rectangle less the area between graph and *x*-axis between $x = 0$ and $x = \ln(2)$.

$$2\ln(2) - \int_0^{\ln(2)} e^x dx = 2\ln(2) - \left[e^x\right]_0^{\ln(2)} = 2\ln(2) - (2-1) = 2\ln(2) - 1$$

or

$x = \ln(y)$

$$\int_1^2 \ln(y)\, dy = \left[y\ln(y) - y\right]_1^2 = 2\ln(2) - 1$$

determines area of rectangle✓

writes correct integral for missing area✓

determines correct answer✓

b

$$\int_0^k e^x dx = 2$$

$\therefore e^k - e^0 = 2$

$e^k = 3$

$k = \ln(3)$

identifies integral to determine area✓

equates integral to 2✓

evaluates integral✓

solves for *k*✓

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Question 9 [SCSA MM2019 Q5] (8 marks)

(✓ = 1 mark)

a First we obtain the area under the graph of $f(x)$ between $x = 0$ and $x = \ln(2)$. This is given by

$$A = \int_0^{\ln(2)} e^x dx = e^x \Big|_0^{\ln(2)} = 2 - 1 = 1$$

writes down the correct integral✓

integrates correctly✓

simplifies to obtain final answer✓

b This is given by the area shown in **Q8aii**.

That is, $Area = 2\ln(2) - 1$

correctly defines the area✓

calculates the area correctly✓

c

$$\int_0^{\ln(a)} e^x dx = e^x \Big|_0^{\ln(a)} = a - 1$$

$$A = a \times \ln(a) - (a - 1)$$
$$= a\ln(a) - a + 1$$

writes down the correct integral✓

integrates correctly and simplifies to obtain *a* − 1✓

determines the correct expression for area✓

Question 10 (4 marks)

(✓ = 1 mark)

a To find the population on 1 January, substitute $t = 0$ in the model.

$N = 500 \ln(21 \times 0 + 3)$

$= 500 \ln(3)$

≈ 549.3

Hence there is a population of **549**✓ moths on 1 January.

b To find the population after 30 days, we substitute $t = 30$ in the model.

$N = 500 \ln(21 \times 30 + 3)$

$= 500 \ln(633)$

≈ 3225.24

Hence there is a population of **3225**✓ months after 30 days.

c For finding the day when population is greater than 2000,

substitute $N = 2000$.

$2000 = 500 \ln(21t + 3)$✓

$\ln(21t + 3) = 4$

Taking exponents on both sides gives us

$21t + 3 = e^4$

$21t = e^4 - 3$

$$t = \frac{e^4 - 3}{21}$$

$$= 2.457\ldots$$

Midnight is the start of the 'date', so there is 1 January (1 day), 2 January (2 days) and into 3 January (0.457 days).

Hence, after **2.457 days, that is 3 January**,✓ they have a population of more than 2000.

Question 11 (8 marks)

(✓ = 1 mark)

a For constructing the model from $T = 60 - a \ln(t + 1)$, we need to find the value of a.

We know after 12 days of intense training, his time was 55 seconds.

At $T = 55$ and $t = 12$,

the equation becomes

$55 = 60 - a \ln(12 + 1)$✓

$5 = a \ln(13)$

$$a = \frac{5}{\ln(13)}$$

$a = 1.949$✓

b $\boldsymbol{T(t) = 60 - 1.949\ln(t+1)}$ ✓

To get the time under 50 seconds, substitute $T = 50$ and solve for t.

$T(t) = 60 - 1.949\ln(t+1)$

$50 = 60 - 1.949\ln(t+1)$

$10 = 1.949\ln(t+1)$

$$\ln(t+1) = \frac{10}{1.949}$$

Taking exponents on both the sides gives us

$$t+1 = e^{\frac{10}{1.949}}$$

$$t = e^{\frac{10}{1.949}} - 1$$

$= 168.1585$

Hence, it will take **169 days**✓ to get the timing less than 50 seconds.

c For finding the rate, we differentiate $T(t)$.

$$\boldsymbol{\frac{dT}{dt} = -\frac{1.949}{t+1}}$$ ✓

at $t = 168.1585$

$$\frac{dT}{dt} = -\frac{1.949}{168.1585+1}$$

$$= -\frac{1.949}{169.1585}$$

≈ -0.012

The negative sign is because of the decrease in the rate.

So the rate is **0.012 seconds per day**.✓

d To get the time under 46 seconds, substitute $T = 46$ and solve for t.

$46 = 60 - 1.949\ln(t+1)$

$14 = 1.949\ln(t+1)$

$$\ln(t+1) = \frac{14}{1.949}$$ ✓

Taking exponents on both the sides gives

$$t = e^{\frac{14}{1.949}} - 1$$

$= 1316.07791\ldots$

≈ 1317 days

He might take 1317 days or 3 years and 7 months.✓

Question 12 (8 marks)

(✓ = 1 mark)

a After 2 weeks he made 7 skateboards per day.

That makes the equation

$7 = 5 + a\ln(2 + 1)$

$a\ln(3) = 2$

$$a = \frac{2}{\ln(3)}$$

$\approx$ **1.820**✓

Hence, the model will be

$\boldsymbol{N = 5 + 1.820\ln(t + 1)}$✓

b For producing 10 skateboards, he will take

$\mathbf{10 = 5 + 1.820\ln(\boldsymbol{t} + 1)}$✓

$1.820\ln(t + 1) = 5$

$$\ln(t+1) = \frac{5}{1.820}$$

Taking exponents on both the sides, we get

$$t+1 = e^{\frac{5}{1.820}}$$

$$t = e^{\frac{5}{1.820}} - 1$$

≈ 14.5997

Which means he will take **15 weeks**✓ to produce 10 skateboards.

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c To find the rate we find the derivative of N with respect to t.

We get

$$\frac{dN}{dt} = \frac{1.820}{t+1} \checkmark$$

So, for rate after 4 weeks, we substitute $t = 4$

$$\frac{dN}{dt} = \frac{1.820}{t+1}$$

$$= \frac{1.820}{4+1}$$

$$= \frac{1.820}{5}$$

$$\approx 0.364$$

Hence the rate after 4 weeks is 0.364 per week.✓

d For the rate after 10 weeks, we substitute $t = 10$, then we get

$$\frac{dN}{dt} = \frac{1.820}{10+1} \checkmark$$

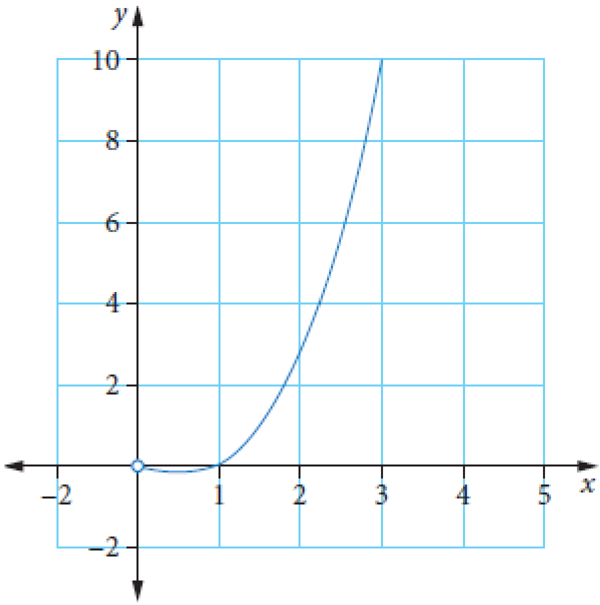
$$\approx 0.165$$

Hence the rate after 10 weeks is 0.165 per week.✓

Question 13 [SCSA MM2016 Q13cd] (5 marks)

(✓ = 1 mark)

a

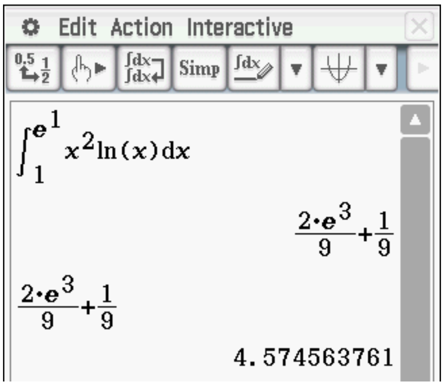


graphs only for $x > 0$✓

one turning point shown between $0 < x < 1$✓

open circle shown at $x = 0$✓

b ClassPad



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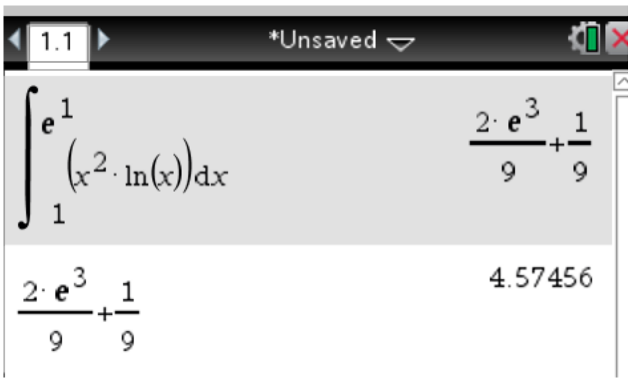

expresses area as a definite integral✓

determines area✓

Question 14 (12 marks)

(✓ = 1 mark)

a **i** We are given $f(x) = \frac{7}{x}$ and know the points $C(1, f(1))$ and $A(a, f(a))$.

The gradient of the line segment CA is $\frac{f(a)-f(1)}{a-1} = \frac{\frac{7}{a}-7}{a-1} = \frac{-\frac{7(a-1)}{a}}{a-1} = -\boldsymbol{\frac{7}{a}}$ ✓

ii The gradient of the function f is given by $f'(x) = -\frac{7}{x^2}$.

We require x such that $f'(x) = -\frac{7}{a}$

$-\frac{7}{x^2} = -\frac{7}{a}, \Rightarrow \boldsymbol{x^2 = a}$ ✓

$x = \pm\sqrt{a}$

(which is defined, as $a > 1$).

As we know $1 < x < a$, only the positive solution is feasible.

Hence $\boldsymbol{x = \sqrt{a}}$. ✓

b **i**

$$\int_1^e f(x)\,dx = 7\int_1^e \frac{1}{x}\,dx$$
$$= 7\left[\log_e(x)\right]_1^e$$
$$= 7\left(\log_e(e) - \log_e(1)\right)$$
$$= \boldsymbol{7}\ ✓$$

ii

$$\int_b^1 f(x)\,dx = 7$$
$$\int_b^1 \frac{7}{x}\,dx = 7$$
$$\int_b^1 \frac{1}{x}\,dx = 1$$
$$\boldsymbol{\left[\log_e(x)\right]_b^1 = 1}\ ✓$$
$$\log_e(1) - \log_e(b) = 1$$
$$\log_e(b) = -1$$
$$\boldsymbol{b = e^{-1}} \text{ or } \boldsymbol{b = \frac{1}{e}}\ ✓$$

c **i** The shape is a trapezium with parallel sides of length 7, $\frac{7}{a}$ and height $a - 1$.

$$Area = \frac{1}{2}\left(7 + \frac{7}{a}\right)(a-1) ✓$$

$$= \frac{1}{2}\left(7a - 7 + 7 - \frac{7}{a}\right)$$

$$= \frac{7a}{2} - \frac{7}{2a}$$

$$= \frac{7(a^2-1)}{2a} ✓$$

Alternatively, Area = Atriangle + Arectangle

$$Area = \frac{1}{2}(a-1)\left(7 - \frac{7}{a}\right) + \frac{7}{a}(a-1) ✓$$

$$= (a-1)\left(\frac{7}{2} - \frac{7}{2a} + \frac{7}{a}\right)$$

$$= (a-1)(a+1)\left(\frac{7}{2a}\right)$$

$$= \frac{7}{2a}(a^2-1) ✓$$

Or, finding the equation of the line segment CA gives

$$Area = \int_1^a \left(-\frac{7}{a}x + \frac{7}{a} + 7\right)dx ✓$$

$$= \frac{7(a^2-1)}{2a} ✓$$

ii We require a such that $\frac{7(a^2-1)}{2a} = 7 \Rightarrow a^2 - 2a - 1 = 0$.

Solving using CAS or the quadratic formula gives $a = 1 \pm \sqrt{2}$.

However, as $a > 0$, the only valid solution is $\boldsymbol{a = 1 + \sqrt{2}}$.✓

iii The curve through CA is below the line segment CA. Hence, the area under the curve is less than the area of the trapezium for $x \in [1, a]$. We know that the area of the trapezium for $a = 1 + \sqrt{2}$ is 7 units^2.

Hence, the area under the curve satisfies the condition $\int_1^a f(x)\,dx < 7$

From part **b i** we know that $\int_1^e f(x)\,dx = 7$ but $\int_1^a f(x)\,dx < 7$, and as f is positive for all positive x, the value of the definite integral increases with the value of the upper bound. So $\boldsymbol{a < e}$.✓

d Evaluating the integral relations given

$$\int_1^{mn} f(x)\,dx = 3$$

$$7\left[\log_e(x)\right]_1^{mn} = 3$$

$$\log_e(mn) - \log_e(1) = \frac{3}{7}$$

$$\log_e(mn) = \frac{3}{7} \qquad [1]$$

$$\int_1^{\frac{m}{n}} f(x)\,dx = 2$$

$$7\left[\log_e(x)\right]_1^{\frac{m}{n}} = 2$$

$$\log_e\left(\frac{m}{n}\right) - \log_e(1) = \frac{2}{7}$$

$$\log_e\left(\frac{m}{n}\right) = \frac{2}{7} \checkmark \qquad [2]$$

Solve these using CAS.

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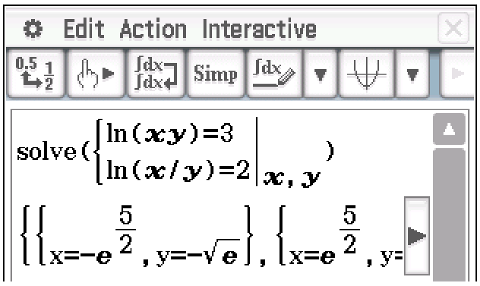
You must replace *m* and *n* with *x* and *y*.



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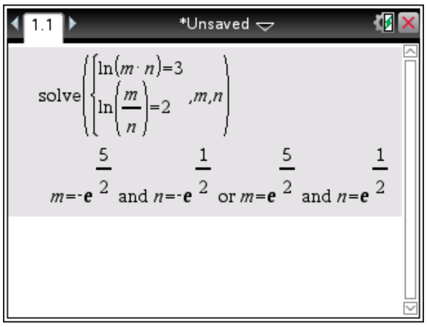

There are two solutions, $\boldsymbol{m = -e^{\frac{5}{14}}, n = -e^{\frac{1}{14}}}$ ✓ and $\boldsymbol{m = e^{\frac{5}{14}}, n = e^{\frac{1}{14}}}$.✓

These solutions can also be obtained by hand.

Note: If you write $\log_e(m)+\log_e(n)=\frac{3}{7}$ and $\log_e(m)-\log_e(n)=\frac{2}{7}$, you will end up with $\log_e(m)=\frac{5}{14}$, $\log_e(n)=\frac{1}{14}$, giving only the positive solution. This is because dividing a logarithm by 2 is equivalent to taking the square root, except that the negative solution does not appear.

An alternative by hand method is to convert the simultaneous logarithmic equations to exponential form, giving $mn = e^{\frac{3}{7}}$ and $\frac{m}{n} = e^{\frac{2}{7}}$. If you multiply the equations, you get $m^2 = e^{\frac{5}{7}} \rightarrow m = \pm e^{\frac{5}{14}}$ which will give both solutions.

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Cumulative examination: Calculator-free

Question 1 (4 marks)

(✓ = 1 mark)

a $t = 0, d = 2$ so $\boldsymbol{d_0 e^0 = 2}$ ✓

$t = 2, d = 10$ so $\boldsymbol{d_0 e^{2m} = 10}$ ✓

b $d_0 e^0 = 2$

$\boldsymbol{d_0 = 2}$ ✓

Hence $2e^{2m} = 10$

$e^{2m} = 5$

$\boldsymbol{m = \frac{1}{2}\ln(5)}$ ✓

Question 2 (7 marks)

(✓ = 1 mark)

a

$$p = 1 - \frac{2}{3}$$

$$= \mathbf{\frac{1}{3}} ✓$$

b

$$\text{P(0 heads)} = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$$

$$\text{P(1 head)} = \binom{3}{1}\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^2 = 3 \times \frac{2}{3} \times \frac{1}{9} = \frac{2}{9}$$

$$\text{P(2 heads)} = \binom{3}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^1 = 3 \times \frac{4}{9} \times \frac{1}{3} = \frac{4}{9}$$

$$\text{P(3 heads)} = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$$

Note: A tree diagram can also be used to find the probabilities.

The probability distribution is

x	0	1	2	3
$P(X = x)$	$\frac{1}{27}$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{8}{27}$

correct values for x used✓

at least two correct values for $P(X = x)$✓

all correct values for $P(X = x)$✓

c

$$P(X \geq 1) = 1 - P(X < 1)$$

$$= 1 - \frac{1}{27}$$

$$= \mathbf{\frac{26}{27}} ✓$$

d

$$\mathbf{P(X = 2 \mid X \geq 1) = \frac{P(X = 2)}{P(X \geq 1)}} ✓$$

$$= \frac{4}{9} \div \frac{26}{27}$$

$$= \mathbf{\frac{6}{13}} ✓$$

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Question 3 (3 marks)

(✓ = 1 mark)

Solve $\left(\mathbf{2e^{2x}-3}\right)\left(\mathbf{e^{x}-2}\right)=\mathbf{0}$ ✓

$2e^{2x}-3=0 \Rightarrow e^{2x}=\frac{3}{2}$ so $\boldsymbol{x=\frac{1}{2}\ln\left(\frac{3}{2}\right)}$ ✓

$e^{x}-2=0 \Rightarrow x=\ln(2)$

The coordinates of the x-intercepts are $\left(\mathbf{\frac{1}{2}\ln\left(\frac{3}{2}\right),0}\right)$ and $\left(\mathbf{\ln(2),0}\right)$.✓

Question 4 [SCSA MM2018 Q3cii] (3 marks)

(✓ = 1 mark)

$$\begin{aligned}\int_0^1 \frac{3x+1}{3x^2+2x+1}\,dx &= \frac{1}{2}\int_0^1 \frac{6x+2}{3x^2+2x+1}\,dx \\ &= \frac{1}{2}\left[\ln\left(3x^2+2x+1\right)\right]_0^1 \\ &= \frac{1}{2}\left[\ln(6)-\ln(1)\right] \\ &= \frac{1}{2}\ln(6)\end{aligned}$$

modifies the integrand so the numerator function is the derivative of the denominator function✓

correctly determines an expression for the integral✓

substitutes and uses log laws to determine a simplified answer✓

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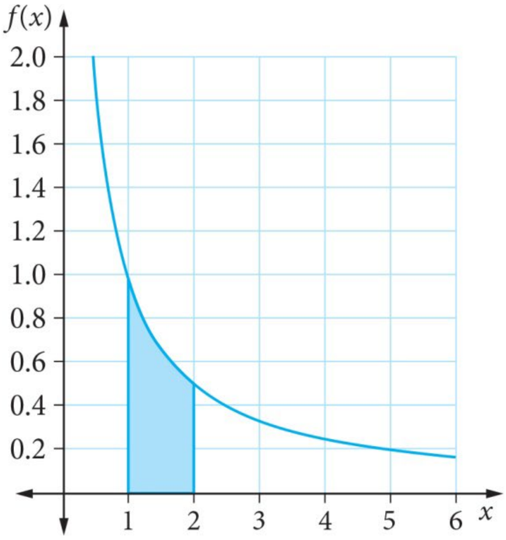
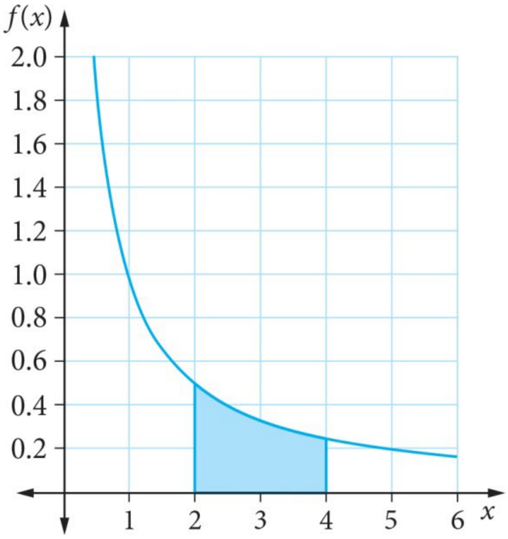
Question 5 [SCSA MM2021 Q7] (9 marks)

(✓ = 1 mark)

a **i**





Two distinct cases in which the upper bound is twice the lower bound. Examples are from $x = 1$ to $x = 2$, and then from $x = 2$ to $x = 4$.

Other possibilities would be $x = 1.5$ to $x = 3$, $x = 2.5$ to $x = 5$ or $x = 3$ to

$x = 6$.

shades a region under the curve corresponding to ln(2)✓

shades a second distinct region under the curve corresponding to ln(2)✓

ii $$\int_a^b \frac{1}{x}\,dx = \ln\left(\frac{b}{a}\right) = \ln(3)$$

So $b = 3a$.

obtains the correct integral in terms of *a* and *b*✓

states the relationship between *a* and *b*✓

b **i** Using the rectangles that estimate $y=\frac{1}{x}$ on the left side of each interval gives

$$\int_2^3 \frac{1}{x}dx < \frac{1}{2}\times\frac{1}{2}+\frac{2}{5}\times\frac{1}{2}=\frac{9}{20}$$

This is an overestimate of the integral as the top of the rectangles lie above the graph.

Using the rectangles that estimate $y=\frac{1}{x}$ on the right side of each interval gives

$$\int_2^3 \frac{1}{x}dx > \frac{2}{5}\times\frac{1}{2}+\frac{1}{3}\times\frac{1}{2}=\frac{11}{30}$$

This is an underestimate of the integral as the top of the rectangles lie below the graph.

Hence,

$$\frac{11}{30} < \int_2^3 \frac{1}{x}dx < \frac{9}{20}$$

approximates the integral using 12×12+25×12=920 ✓

approximates the integral using 25×12+13×12=1130 ✓

explains why the first approximation is an overestimate and the second is an underestimate✓

ii $$\int_2^3 \frac{1}{x}dx = \left[\ln(x)\right]_2^3 = \ln(3)-\ln(2)=\ln(1.5)$$

Hence

$$\frac{11}{30} < \ln(1.5) < \frac{9}{20}$$

correctly integrates $\frac{1}{x}$ **and substitutes bounds to obtain ln(3) – ln(2)**✓

applies log law to obtain ln(3) – ln(2) = ln(1.5)✓

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Question 6 (3 marks)

(✓ = 1 mark)

$f'(x)=\frac{1}{2}-\frac{1}{2x-2}$ and $f(2)=0$

$\boldsymbol{f(x)=\frac{x}{2}-\frac{1}{2}\log_e(2x-2)+c}$ ✓

Given that $f(2)=0$,

$0=\frac{2}{2}-\frac{1}{2}\log_e(2\times2-2)+c$

$0=1-\frac{1}{2}\log_e(2)+c$

$\boldsymbol{c=-1+\frac{1}{2}\log_e(2)}$ ✓

Substituting $c=-1+\frac{1}{2}\log_e(2)$ into $f(x)$, we get

$\boldsymbol{f(x)=\frac{x}{2}-\frac{1}{2}\log_e(2x-2)-1+\frac{1}{2}\log_e(2)}$ ✓

Question 7 (7 marks)

(✓ = 1 mark)

a $f(x)=x^2e^{kx}$

Use the product rule $\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$

Let $u=x^2$ $\qquad \frac{du}{dx}=2x$

Let $v=e^{kx}$ $\qquad \frac{dv}{dx}=ke^{kx}$

$f'(x)=\boldsymbol{x^2\times ke^{kx}+e^{kx}\times2x}$ ✓

$=xe^{kx}(kx+2)$

b Solve $f(x)=f'(x)$

$$x^2 \cancel{e^{kx}} = x\cancel{e^{kx}}(kx+2) \text{ since } e^{kx}\neq 0$$

$$x^2 = kx^2+2x$$

$$x^2-kx^2-2x=0$$

$$x(x-kx-2)=0$$

$$\boldsymbol{x=0 \text{ or } x-kx-2=0} \checkmark$$

$$x-kx-2=0$$

$$x(1-k)=2$$

$$x=\frac{2}{1-k}$$

There will be exactly one point of intersection at (0, 0) because $x=\dfrac{2}{1-k}$ is undefined when $\boldsymbol{k=1}\checkmark$

c

$$f(x)=x^2e^{kx} \text{ and } g(x)=-\frac{2xe^{kx}}{k}$$

$$A=\int_0^2(f(x)-g(x))\,dx=\int_0^2\left(x^2e^{kx}+\frac{2xe^{kx}}{k}\right)dx \checkmark$$

d

$$\int_0^2\left(x^2e^{kx}+\frac{2}{k}xe^{kx}\right)dx=\frac{16}{k}$$

$$\frac{1}{k}\int_0^2\left(kx^2e^{kx}+2xe^{kx}\right)dx=\frac{16}{k}$$

$$\frac{1}{k}\int_0^2\left(xe^{kx}(kx+2)\right)dx=\frac{16}{k}$$

$$\frac{1}{\cancel{k}}\left[x^2e^{kx}\right]_0^2=\frac{16}{\cancel{k}}\checkmark$$

From part **a**, $\dfrac{d}{dx}\left(x^2e^{kx}\right)=xe^{kx}(kx+2)$

$$2^2e^{k\times 2}-0^2e^{0\times k}=16$$

$$4e^{2k}=16$$

$$\boldsymbol{e^{2k}=4}\checkmark$$

$$2k=\log_e(4)$$

$$k=\frac{1}{2}\log_e(4)$$

$$\boldsymbol{k=\log_e(2)=\ln(2)}\checkmark$$

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Cumulative examination: Calculator-assumed

Question 1 (10 marks)

(✓ = 1 mark)

a **Stationary points when $f'(x)=0$.** ✓

Using algebra.

Use product rule.

$$f'(x)=\frac{1}{27}\left((ax-1)^3\frac{d}{dx}(b-3x)+(b-3x)\frac{d}{dx}\left((ax-1)^3\right)\right)$$

$$=\frac{1}{27}\left(-3(ax-1)^3+3a(b-3x)(ax-1)^2\right)$$

$$=\frac{1}{27}\left(3(ax-1)^2\left(-(ax-1)+a(b-3x)\right)\right)$$

$$=\frac{1}{27}\left(3(ax-1)^2(-ax+1+ab-3ax)\right)$$

$$=-\frac{1}{9}(ax-1)^2(4ax-ab-1)$$

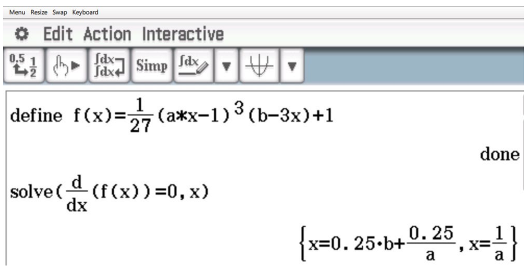
$$-\frac{1}{9}(ax-1)^2(4ax-ab-1)=0$$

$(ax-1)^2=0$ so $\boldsymbol{x=\frac{1}{a}}$ ✓

$4ax-ab-1=0$ so $\boldsymbol{x=\frac{ab+1}{4a}}$ ✓

Or using CAS:

ClassPad



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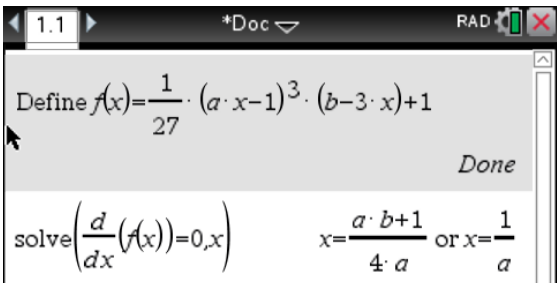


b $\frac{1}{a}$ is not defined for $a = 0$.

Hence there is no stationary point when $\boldsymbol{a = 0}$.✓

c Let $\frac{\mathbf{1}}{\boldsymbol{a}} = \frac{\boldsymbol{ab}+\mathbf{1}}{\mathbf{4}\boldsymbol{a}}$ ✓

$1 = \frac{ab+1}{4}$, since a is not zero.

$ab+1=4$

$\boldsymbol{a} = \frac{\mathbf{3}}{\boldsymbol{b}}$ ✓

d Stationary points at $x = \frac{1}{a}$ and $x = \frac{ab+1}{4a}$. Since $a \neq 0$, the **maximum number of stationary points is 2**.✓

e (1, 1) means there is a stationary point at $x = 1$.

Hence $1 = \frac{1}{a} \Rightarrow a = 1$ ✓

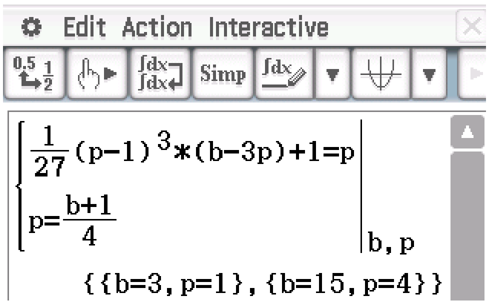
(p, p) means there is a stationary point at $x = p$.

Hence $p = \frac{ab+1}{4a} = \frac{b+1}{4}$.✓

$f(p) = \frac{1}{27}(p-1)^3(b-3p)+1 = p$

Use CAS to solve for b and p.

ClassPad



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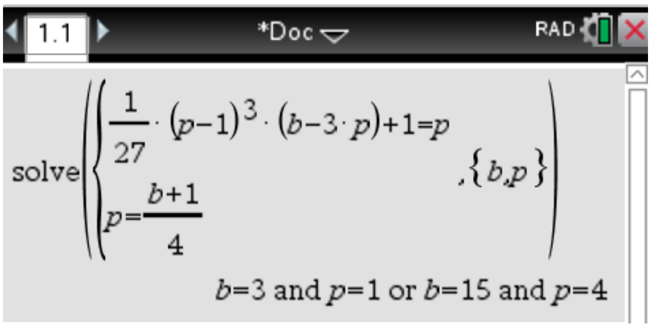

Since $p \neq 1$, then $\boldsymbol{p = 4}$.✓

Question 2 (1 mark)

(✓ = 1 mark)

$$\int_1^{12} g(x)\,dx + \int_{12}^{5} g(x)\,dx = \int_1^5 g(x)\,dx$$
$$= 5 + (-6)$$
$$= \mathbf{-1} ✓$$

Question 3 (9 marks)

(✓ = 1 mark)

a

$$f'(x) = x^4 \frac{d}{dx}(\ln(4x)) + \frac{d}{dx}(x^4)\ln(4x) ✓$$
$$= x^4 \times \frac{1}{x} + 4x^3 \ln(4x)$$
$$= \mathbf{x^3 + 4x^3 \ln(4x)} \text{ or } \mathbf{x^3(1 + 4\ln(4x))} ✓$$

b

$$\frac{d}{dx}(x^4 \ln(4x)) = x^3 + 4x^3 \ln(4x) ✓$$
$$4\int x^3 \ln(4x)\,dx = \int \frac{d}{dx}(x^4 \ln(4x))dx - \int x^3 dx ✓$$
$$\int x^3 \ln(4x)\,dx = \frac{1}{4}\left(x^4 \ln(4x) - \frac{1}{4}x^4\right)$$
$$\int x^3 \ln(4x)\,dx = \frac{1}{4}x^4 \ln(4x) - \frac{1}{16}x^4 + c ✓$$

c

$$\int_{0.25}^{1} x^3 \ln(4x)\,dx = \left[\frac{1}{4}x^4 \ln(4x) - \frac{1}{16}x^4\right]_{0.25}^{1} ✓$$
$$= \left(\frac{1}{4}1^4 \ln(4) - \frac{1}{16}1^4\right) - \left(\frac{1}{4}0.25^4 \ln(1) - \frac{1}{16}0.25^4\right)$$
$$= \frac{1}{4} \times 2\ln(2) - \frac{1}{16} + \frac{1}{4096}$$
$$= \frac{1}{2}\ln(2) - \frac{255}{4096} ✓$$

d

$$\int_{0.25}^{1} t^3 \ln(4t)\,dt = \frac{1}{2}\ln(2) - \frac{255}{4096} ✓$$
$$= 0.2843\ldots \text{ m}$$
$$= \mathbf{28 \text{ cm}} ✓$$

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Question 4 [SCSA MM2020 Q11] (9 marks)

(✓ = 1 mark)

a $f'(x) = e^x = 1$

For $x = 0, f(0) = 1$

So the point of intersection is (0, 1).

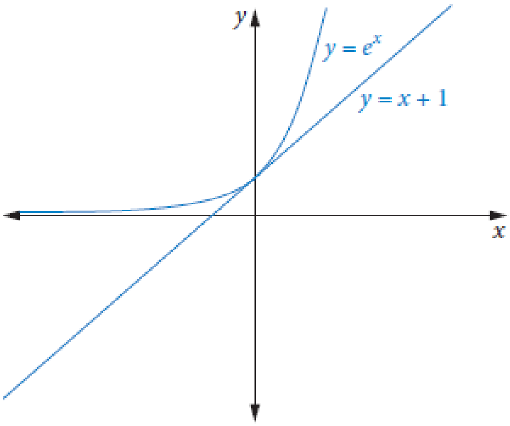
obtains equation to solve for the x coordinate✓

states the coordinates of the point✓

b The point (0, 1) lies on the line, so $1 = 0 + c \Rightarrow c = 1$

obtains the correct value of c✓

c



sketches both functions showing tangent at (0, 1) with shapes correct✓

d $$Area = \int_0^{\ln 2} \left(e^x - (x+1)\right) dx$$

$$= \left[e^x - \frac{x^2}{2} - x \right]_0^{\ln 2}$$

$$= 2 - \frac{(\ln(2))^2}{2} - \ln(2) - 1$$

$$= 1 - \frac{(\ln(2))^2}{2} - \ln(2)$$

writes down the correct integrand✓

gives the correct limits for the integral✓

evaluates correctly✓

e $$Area = -\int_{\ln 2}^{1} \ln(x)\, dx$$

recognises the inverse function of $f(x)$✓

states the correct definite integral✓

Chapter 8 – Continuous random variables and the normal distribution

EXERCISE 8.1 General continuous random variables

Question 1

a For each, read the value of the height of the corresponding rectangle.

$\boldsymbol{n(20 \le t < 25) = 3,\ n(35 \le t < 40) = 6}$

b **i** $\mathrm{P}(15 \le T \le 20) = \dfrac{2}{28} = \mathbf{\dfrac{1}{14}}$

ii $$\mathrm{P}(T \ge 15 \mid T \le 20) = \frac{\mathrm{P}(15 \le T \le 20)}{\mathrm{P}(T < 20)}$$

$$= \frac{\frac{2}{28}}{\frac{4+1+1+2}{28}}$$

$$= \mathbf{\frac{1}{4}}$$

iii $$\mathrm{P}(T < 40) = \frac{4+1+1+2+3+1+2+6}{28}$$

$$= \mathbf{\frac{5}{7}}$$

iv $$\mathrm{P}(T \ge 20) = 1 - \mathrm{P}(T < 20) = 1 - \frac{4+1+1+2}{28}$$

$$= \mathbf{\frac{5}{7}}$$

v $42 \le 48$ crosses 2 of the given ranges, being $\dfrac{3}{5}$ of $40 \le t < 45$ and $\dfrac{3}{5}$ of $45 \le t < 50$.

Calculated frequency for the required region = $\dfrac{3}{5}(3) + \dfrac{3}{5}(2) = \dfrac{15}{5} = 3$.

So $\mathrm{P}(42 \le T \le 48) = \mathbf{\dfrac{3}{28}}$

Question 2

Determine the two rules that define the hybrid function and their domains.
The probability density function is

$$\int_0^5 \left(k\left(5x - x^2\right)\right) dx = 1$$

$$k\left(\frac{5}{2}x^2 - \frac{1}{3}x^3\right)\Bigg|_0^5 = 1$$

$$k\left(\frac{125}{2} - \frac{125}{3}\right) = 1$$

$$\frac{125}{6}k = 1$$

$$\boldsymbol{k = \frac{6}{125}}$$

Question 3

a Since $\text{P}(X = k) = 0$, $\mathbf{P(X = 2) = 0}$.

b The gradient of the line is $\frac{2}{3} \div 3 = \frac{2}{9}$ and the equation is $y = \frac{2}{9}x$.

$$\text{P}(0 \le X \le 2) = \int_0^2 \frac{2}{9}x\, dx$$

$$= \left[\frac{1}{9}x^2\right]_0^2$$

$$\mathbf{= \frac{4}{9}}$$

c

$$\text{P}(1 < X < 2) = \int_1^2 \frac{2}{9}x\, dx$$

$$= \left[\frac{1}{9}x^2\right]_1^2 = \frac{4}{9} - \frac{1}{9}$$

$$\mathbf{= \frac{1}{3}}$$

d

$$\text{P}(X > 1 \mid X < 2) = \frac{\text{P}(1 < X < 2)}{\text{P}(X < 2)}$$

$$= \frac{1}{3} \div \frac{4}{9}$$

$$\mathbf{= \frac{3}{4}}$$

Question 4

a

$$\mathrm{P}\left(X \geq \frac{1}{2}\right) = 1 - \mathrm{P}\left(X < \frac{1}{2}\right)$$

$$= 1 - \int_0^{\frac{1}{2}} 4x^3 dx$$

$$= 1 - \left[\frac{4x^4}{4}\right]_0^{\frac{1}{2}}$$

$$= 1 - \left[\frac{4\left(\frac{1}{2}\right)^4}{4} - \frac{4(0)^4}{4}\right]$$

$$= 1 - \frac{1}{16}$$

$$= \mathbf{\frac{15}{16}}$$

b

$$\int_{50}^{80} 0.01e^{-0.01y} dy$$

$$= \left[-e^{-0.01y}\right]_{50}^{80}$$

$$= -e^{-0.8} + e^{-0.5}$$

$$= \frac{1}{e^{\frac{1}{2}}} - \frac{1}{e^{\frac{4}{5}}}$$

$$= \mathbf{\frac{1}{\sqrt{e}} - \frac{1}{\sqrt[5]{e^4}}}$$

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c

$$P\left(Z > \frac{1}{4} \mid Z < \frac{1}{3}\right) = \frac{P\left(\frac{1}{4} < Z < \frac{1}{3}\right)}{P\left(Z < \frac{1}{3}\right)}$$

$$P\left(\frac{1}{4} < Z < \frac{1}{3}\right)$$

$$= \int_{\frac{1}{4}}^{\frac{1}{3}} \left(\pi \sin(2\pi z)\right) dz$$

$$= \left[\frac{-\pi\left(\cos(2\pi z)\right)}{2\pi}\right]_{\frac{1}{4}}^{\frac{1}{3}}$$

$$= \left[\frac{-\pi\left(\cos\left(2\pi \times \frac{1}{3}\right)\right)}{2\pi}\right] - \left[\frac{-\pi\left(\cos\left(2\pi \times \frac{1}{4}\right)\right)}{2\pi}\right]$$

$$= \frac{1}{4} - 0 = \frac{1}{4}$$

Find $P\left(Z < \frac{1}{3}\right)$.

$$= \int_{0}^{\frac{1}{3}} \left(\pi \sin(2\pi z)\right) dz$$

$$= \left[\frac{-\pi\left(\cos(2\pi z)\right)}{2\pi}\right]_{0}^{\frac{1}{3}}$$

$$= \left[\frac{-\pi\left(\cos\left(2\pi \times \frac{1}{3}\right)\right)}{2\pi}\right] - \left[\frac{-\pi\left(\cos(2\pi \times 0)\right)}{2\pi}\right]$$

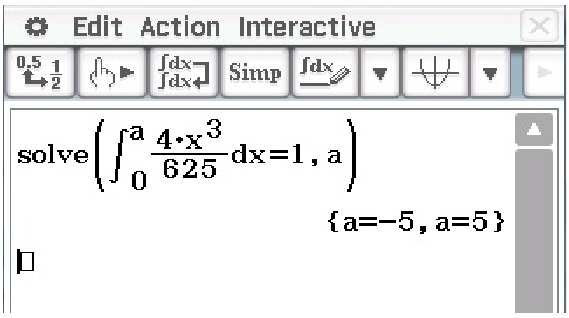
$$= \frac{1}{4} + \frac{1}{2}$$

$$= \frac{3}{4}$$

Hence $P\left(Z > \frac{1}{4} \mid Z < \frac{1}{3}\right) = \frac{\frac{1}{4}}{\frac{3}{4}} = \mathbf{\frac{1}{3}}$

Question 5

ClassPad



1 Enter the definite integral with lower and upper limits of **0** and **a** and set equal to **1**.

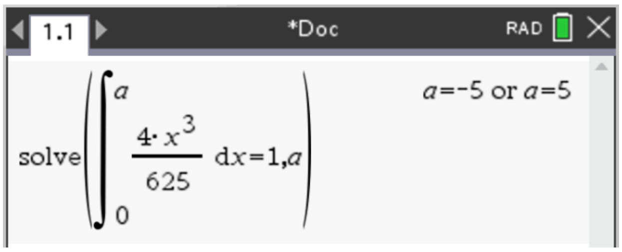
2 Highlight the integral equation and solve for a, as shown above.

3 Select the positive solution.

$a \neq -5$ as this is outside the domain.

$\boldsymbol{a = 5}$

TI-Nspire



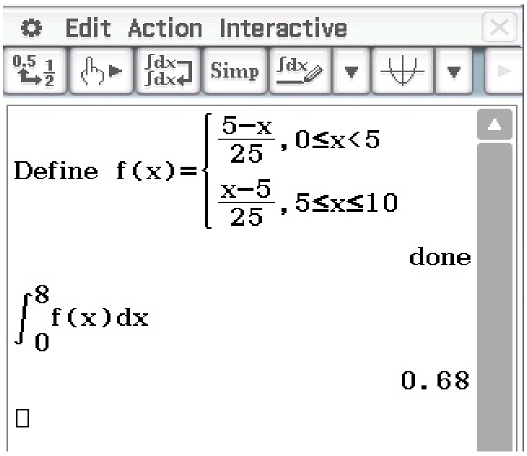
1 Set the definite integral equal to 1 and solve for a, as shown above.

2 Select the positive solution.

Question 6

ClassPad

a

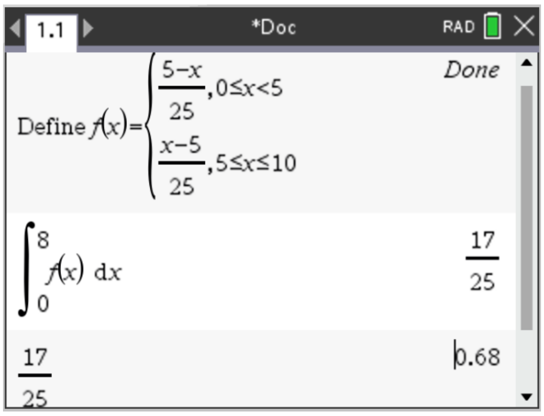


1 Enter and define the function **f(x)** using the **Math3** piecewise template.

2 Find the definite integral of **f(x)** from 0 to 8.

$\Pr(X < 8) = \mathbf{0.68}$.

TI-Nspire



1 Define the piecewise function **f(x)**.

2 Find the definite integral of **f(x)** from 0 to 8.

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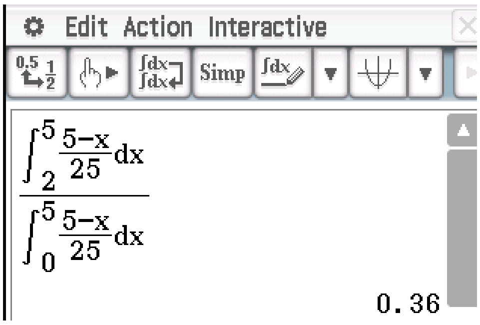
b

$$\mathrm{P}(X \geq 2 \mid X < 5) = \frac{\mathrm{P}(2 \leq X < 5)}{\mathrm{P}(X < 5)}$$

$$= \frac{\int_2^5 \frac{5-x}{25}\,dx}{\int_0^5 \frac{5-x}{25}\,dx} = \frac{0.18}{0.5}$$

=0.36

ClassPad



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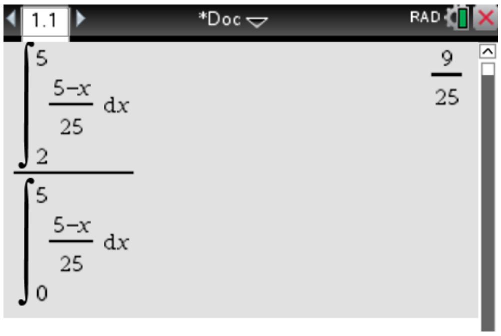
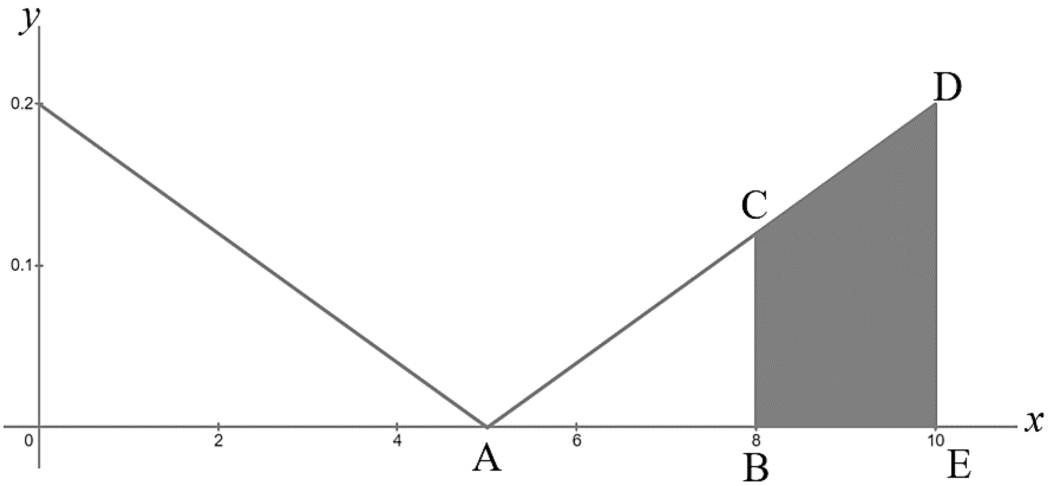


c



$$\mathrm{P}(X > 8 \mid X > 5) = \frac{\mathrm{P}(X > 8)}{\mathrm{P}(X > 5)}$$

$$= \frac{\int_8^{10} \frac{x-5}{25}\,dx}{\int_5^{10} \frac{x-5}{25}\,dx} = \mathbf{0.64}$$

or

$$\mathrm{P}(X > 8 \mid X > 5) = \frac{\text{area BCDE}}{\text{area AED}}$$

$$= \frac{\frac{1}{2}\left(\frac{3}{25} + \frac{5}{25}\right) \times 2}{\frac{1}{2} \times 5 \times \frac{5}{25}}$$

$$= \mathbf{\frac{16}{25} = 0.64}$$

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Question 7

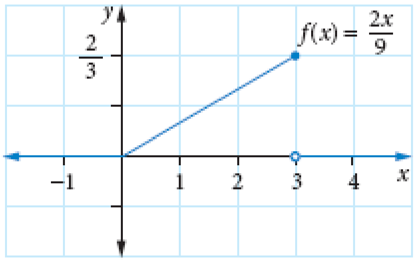
Determine the two rules that define the hybrid function and their domains.

The probability density function is

$$f(x)=\frac{2}{9}x,\quad 0\le x\le 3p$$

$$f(x)=0,\ x<0 \text{ and } x>3$$

Sketch the graphs over the three domains.



a Write in the form

$$F(x)=\begin{cases}0 & x<a\\ \int_a^x f(t)\,dt & a\le x\le b\\ 1 & x>b\end{cases},\ \text{with } a=0,\ b=3$$

$$\int_a^x f(t)\,dt=\int_0^x \frac{2t}{9}\,dt=\left[\frac{t^2}{9}\right]_0^x=\frac{x^2}{9}$$

Hence

$$\boldsymbol{F(x)=\begin{cases}0 & x<0\\ \dfrac{x^2}{9} & 0\le x\le 3\\ 1 & x>3\end{cases}}$$

b

$$\text{P}\left(X\le\frac{5}{2}\right)=\left[\frac{x^2}{9}\right]_0^{\frac{5}{2}}$$

$$=\frac{\left(\frac{5}{2}\right)^2}{9}$$

$$=\mathbf{\frac{25}{36}}$$

Question 8

a

$$P(X \geq 6) = \left[\frac{x^2}{32} - \frac{x}{4} + 1\right]_6^8$$

$$= \left(\frac{8^2}{32} - \frac{8}{4} + 1\right) - \left(\frac{6^2}{32} - \frac{6}{4} + 1\right)$$

$$= 1 - 0.625$$

$$\mathbf{= 0.375 = \frac{3}{8}}$$

b The probability density function of a continuous random variable X is the derivative of the cumulative distribution function.

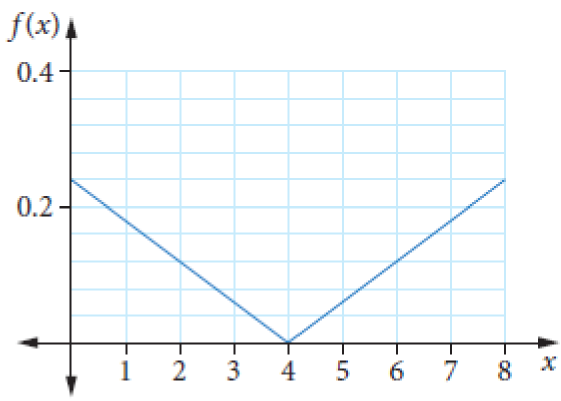
$$f(x) = \frac{d}{dx}\left(\frac{x}{4} - \frac{x^2}{32}\right) = \frac{1}{4} - \frac{x}{16}, \quad 0 \leq x < 4$$

$$f(x) = \frac{d}{dx}\left(\frac{x^2}{32} - \frac{x}{4} + 1\right) = \frac{x}{16} - \frac{1}{4}, \quad 4 \leq x \leq 8$$

Hence the cumulative distribution function is

$$\boldsymbol{f(x) = \begin{cases} \frac{1}{4} - \frac{x}{16} & 0 \leq x < 4 \\ \frac{x}{16} - \frac{1}{4} & 4 \leq x \leq 8 \\ 0 & \text{otherwise} \end{cases}}$$

c



d $\int_0^4 \left(\frac{1}{4}-\frac{x}{16}\right) dx + \int_4^x \left(\frac{t}{16}-\frac{1}{4}\right) dt$

$= \frac{1}{2} + \left[\frac{t^2}{32}-\frac{t}{4}\right]_4^x$

$= \frac{1}{2} + \frac{x^2}{32} - \frac{x}{4} - \frac{1}{2} + 1$

$= \frac{x^2}{32} - \frac{x}{4} + 1$

Question 9 [SCSA MM2017 Q1] (5 marks)

(✓ = 1 mark)

a $P(T \le 36) = 0.02 + 0.04 + 0.04$

$= 0.1$

sums relative frequencies to determine probability✓

b $P(35 \le T \le 39) = 0.04 + 0.02 + 0.08 + 0.12$

$= 0.26$

recognises the probability involves frequencies above 35 and below 39✓

sums relative frequencies to determine probability✓

c $P(T \le 38) = 0.02 + 0.04 + 0.04 + 0.02 + 0.08$

$= 0.2$

3 consecutive days = $0.2^3 = 0.008$

sums relative frequencies to determine probability✓

determines probability of 3 consecutive days✓

Question 10 **[SCSA MM2020 Q4] (9 marks)**

(✓ = 1 mark)

a $P(h \le 30) = 0.04 + 0.08$

$= 0.12$

determines the correct probability✓

b $P(h \ge 38 \mid h \ge 32) = \dfrac{0.02}{0.56} = \dfrac{1}{28}$

recognises conditional probability and determines the correct denominator of the conditional probability✓

determines the correct probability as a fraction✓

c Probability of $1 < h < 2$:

$$\int_1^2 \frac{h-1}{5}\,dh = \frac{1}{5}\left[\frac{h^2}{2} - h\right]_1^2$$

$$= \frac{1}{5}\left[2 - 2 - \frac{1}{2} + 1\right]$$

$$= \frac{1}{10}$$

10% reach a height less than 2 m

recognises the need to integrate the first equation of the pdf from 1 to 2✓

anti-differentiates the first equation correctly✓

determines the correct percentage✓

d Probability of $2 < h \le 4 = 1 - \dfrac{1}{10} = \dfrac{9}{10}$

$$\therefore \frac{9}{10} = \int_2^4 kh^2\,dh$$

$$\frac{9}{10} = k\left[\frac{h^3}{3}\right]_2^4$$

$$\frac{9}{10} = k\left[\frac{64-8}{3}\right]$$

$$k = \frac{27}{560}$$

recognises the need to integrate the second equation of the pdf from 2 to 4 and equates to the complement of part c, $\dfrac{9}{10}$ ✓

anti-differentiates the second equation correctly✓

determines the value of k✓

Question 11 **(7 marks)**

(✓ = 1 mark)

a The area under the curve in $0 \le x \le 4$ is 1.

$$\int_0^4 \frac{x+1}{k} dx = 1 \checkmark$$

$$\left[\frac{1}{k}\left(\frac{1}{2}x^2 + x\right)\right]_0^4 = 1$$

$$\frac{1}{k}\left(8+4-[0+0]\right) = 1$$

$$k = 12 \checkmark$$

b The area under the curve in $0 \le x \le 4$ is $\frac{5}{8}$.

$$\int_0^b \frac{x+1}{12} dx = \frac{5}{8} \checkmark$$

$$\left[\frac{1}{12}\left(\frac{1}{2}x^2 + x\right)\right]_0^b = \frac{5}{8}$$

$$\frac{1}{12}\left(\frac{1}{2}b^2 + b\right) = \frac{5}{8}$$

$$2\left(\frac{1}{2}b^2 + b\right) = 15$$

$$b^2 + 2b - 15 = 0$$

$$(b-3)(b+5) = 0$$

$$b = 3, b = -5 \checkmark$$

Since $b > 0$, take $b = 3$.✓

c

$$P(X > 2 \mid X < 3) = \frac{P(2 < X < 3)}{P(X < 3)} \checkmark$$

$$P(2 < X < 3) = \int_2^3 \frac{x+1}{12} dx = \left[\frac{1}{12}\left(\frac{1}{2}x^2 + x\right)\right]_2^3 = \frac{1}{12}\left(7\frac{1}{2} - 4\right) = \frac{7}{24}$$

$\int_0^3 \frac{x+1}{12} dx = \frac{5}{8}$, from part b.

$$P(X > 2 \mid X < 3) = \frac{\frac{7}{24}}{\frac{5}{8}} = \frac{7}{15} \checkmark$$

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Question 12 (4 marks)

(✓ = 1 mark)

a If $f(x)$ is a probability density function, it must have two properties: $f(x) \geq 0$ for all $x \in R$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$ (The area enclosed by its curve and the x-axis is 1).

$\boldsymbol{\frac{a}{x^2} > 0}$ **because** $\boldsymbol{a > 0}$ **and** $\boldsymbol{x^2 > 0,\ f(x) \geq 0}$ **for all** $\boldsymbol{x \in R}$✓

$$\lim_{k\to\infty}\left(\int_a^k \frac{a}{x^2}\,dx\right) = \lim_{k\to\infty}\left(\left[-\frac{a}{x}\right]_a^k\right) = \lim_{k\to\infty}\left(-\frac{a}{k}+1\right)$$

As $k \to \infty$, $-\frac{a}{k} + 1 \to 1$. ✓

As $f(x)$ satisfies both properties, it is a **valid probability distribution function**.

b $P(X > 2a) = 1 - P(X \leq 2a)$

$$= 1 - \int_a^{2a} \frac{a}{x^2}\,dx \ ✓$$

$$= 1 - \left[-\frac{a}{x}\right]_a^{2a}$$

$$= 1 - \left(-\frac{a}{2a} - \left[-\frac{a}{a}\right]\right)$$

$$= 1 - \left(-\frac{1}{2} + 1\right)$$

$$= \mathbf{\frac{1}{2}} ✓$$

Question 13 [SCSA MM2017 Q11a] (2 marks)

(✓ = 1 mark) ✓

$$P(X < 0.5) = \int_0^{0.5}\left(\frac{4}{3} - \frac{2}{3}x\right)dx = \frac{4}{3}x - \frac{1}{3}x^2\Big|_0^{0.5} = \frac{2}{3} - \frac{1}{12} = \frac{7}{12}$$

or

$$P(X < 0.5) = \textbf{Area of trapezium} = \ = \frac{\frac{4}{3}+1}{2} \times \frac{1}{2} = \frac{7}{12}$$

writes correct integral (or area) expression for probability✓

calculates probability correctly✓

Question 14 **[SCSA MM2018 Q10ab] (5 marks)**

(✓ = 1 mark)

a

If pdf on domain then $\int_0^2 f(x)\,dx = 1$

$$\int_0^2 ax^2(x-2)\,dx = -\frac{4a}{3}$$

$$\therefore -\frac{4a}{3} = 1$$

$$a = -\frac{3}{4}$$

uses integration for domain✓

calculates integration✓

finds *a*✓

b

$$\int_{1.2}^{2} \frac{-3x^2(x-2)}{4}\,dx = 0.5248$$

uses correct integral✓

calculates probability✓

Question 15 (7 marks)

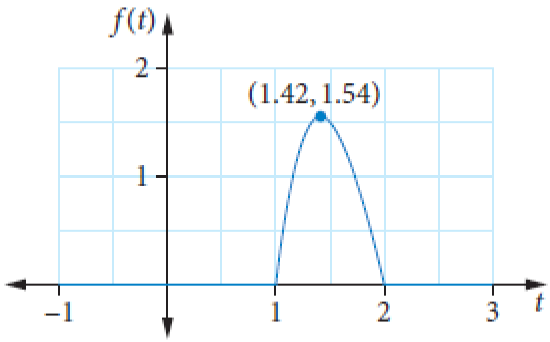
(✓ = 1 mark)

a As the stationary points are to be expressed correct to two decimal places, you might suspect that they are irrational, which would suggest that using CAS would be quicker and less prone to errors than a by-hand solution.

Sketch the graph on your CAS (as a function of x for $1 \le x \le 2$) and calculate the turning point(s). You may need to adjust the zoom/window to display the graph clearly.

Alternatively, define $f(t)$ and use your CAS to solve $f(t) = 0$ and $f'(t) = 0$.

Substitute the values of any solutions into $f(t)$.



correct turning point with coordinates shown✓

graph drawn in interval [1, 2]✓

general shape accurate, including axes labels.✓

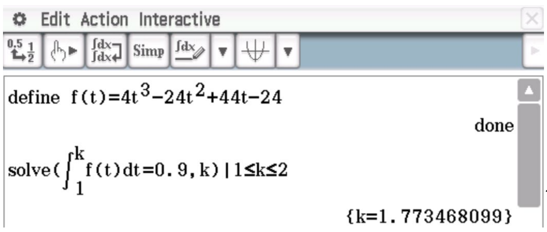
b The probability that Kim spends less than 75 minutes (1.25 hours) working out is

$$\int_1^{\frac{5}{4}} f(t)\,dt = \int_1^{\frac{5}{4}} \left(4t^3 - 24t^2 + 44t - 24\right) dt ✓$$

Using CAS (set to Approximate or Decimal) this integral is 0.1914…, so correct to three decimal places, the probability that Kim spends less than 45 minutes working out is **0.191**✓

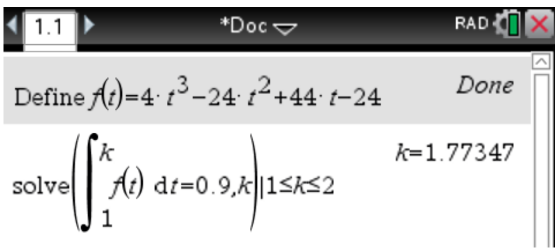
c P(more than k minutes) = 0.1 is 1 – P(less than k minutes)

ClassPad



✓

TI-Nspire

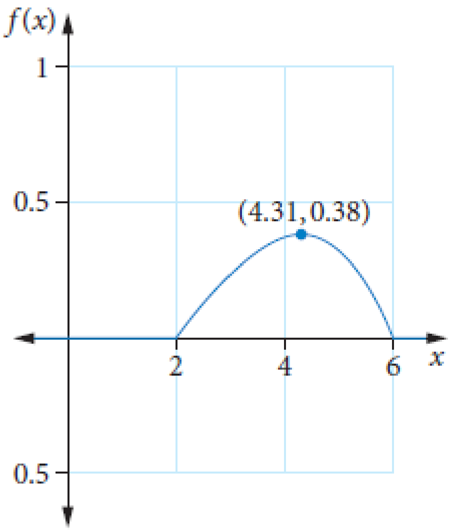


1.7734 hours is 106 minutes✓

Question 16 (4 marks)

(✓ = 1 mark)

a



correct turning point with coordinates shown✓

graph drawn in interval [2, 6]✓

b

$$\int_2^3 \frac{1}{64}(6-x)(x-2)(x+2)\,dx \approx 0.1211$$✓

The probability is 0.1211✓

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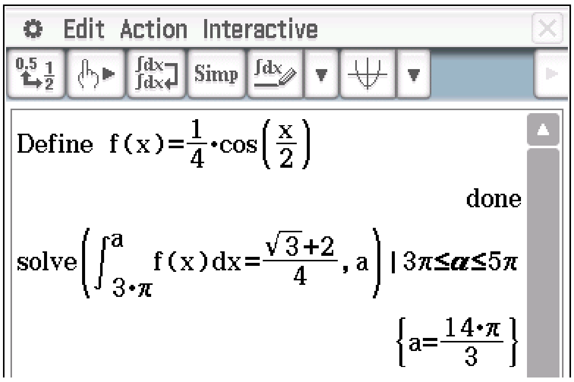
Question 17 (6 marks)

(✓ = 1 mark)

a Define $f(x)=\frac{1}{4}\cos\left(\frac{x}{2}\right)$ and solve $\int_{3\pi}^{a} f(x)\,dx=\frac{\sqrt{3}+2}{4}$ to determine the value of a.

ClassPad



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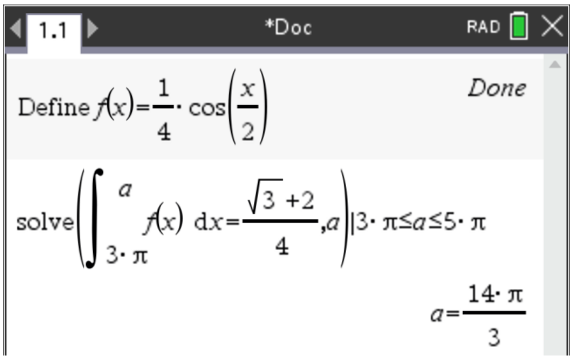


$a=\frac{\mathbf{14\pi}}{\mathbf{3}}=\mathbf{14.66}$✓

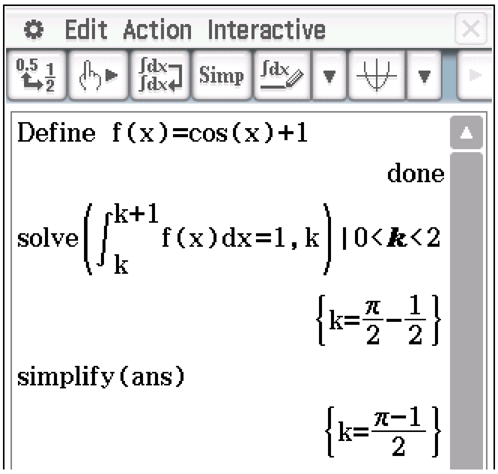
Use correct integral✓

Use correct equation to solve✓

b Calculate the integral of $f(t)=\cos(t)+1$ from k to $k+1$.

Set the integral equal to 1 and solve for $0<k<2$.

ClassPad



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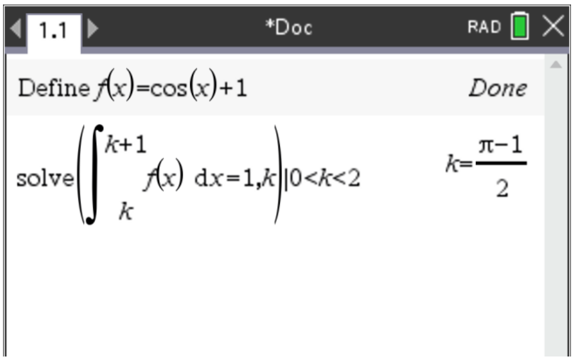

$\boldsymbol{k}=\frac{\boldsymbol{\pi}-\mathbf{1}}{\mathbf{2}}$ ✓

use correct integral✓

use correct equation to solve✓

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Question 18 (7 marks)

(✓ = 1 mark)

a $\int_0^4 \frac{y}{16} dy + \lim_{k \to \infty} \left(\int_4^k 0.25e^{-0.5(y-4)} dy \right)$ ✓

$= \left[\frac{y^2}{32} \right]_0^4 + \lim_{k \to \infty} \left(\left[-0.5e^{-0.5(y-4)} \right]_4^k \right)$ ✓

$= 0.5 + \lim_{k \to \infty} \left(-0.5e^{-0.5(k-4)} + 0.5e^0 \right)$

$= 0.5 + \lim_{k \to \infty} \left(-0.5e^{-0.5(k-4)} \right) + \lim_{k \to \infty} \left(0.5e^0 \right)$

$= 0.5 + 0 + 0.5 = \mathbf{1}$ ✓

All values of *f* (*y*) are positive and sum of all probabilities equals 1. Therefore, a valid probability density function.✓

b $P(3 \le Y \le 5) = \int_3^5 f(y) dy$

$= \int_3^4 \frac{y}{16} dy + \int_4^5 0.25e^{-0.5y+2} dy$ ✓

$= \frac{1}{32} \left[y^2 \right]_3^4 - \frac{1}{2} \left[e^{-0.5y+2} \right]_4^5$ ✓

$= \frac{1}{32}(16-9) - \frac{1}{2}\left(e^{-0.5} - e^0\right)$

$= \frac{7}{32} + \frac{1}{2}\left(1 - e^{-0.5}\right)$

$\approx \mathbf{0.4155}$ ✓

EXERCISE 8.2 Measures of centre and spread

Question 1

Express the probability as a definite integral.

$$\begin{aligned}
P\left(X > \frac{1}{2}\right) &= 1 - P\left(X \le \frac{1}{2}\right) \\
&= 1 - \int_0^{\frac{1}{2}} 3x^2 dx \\
&= 1 - \left[x^3\right]_0^{\frac{1}{2}} \\
&= 1 - \left(\left(\frac{1}{2}\right)^3 - (0)^3\right) \\
&= 1 - \frac{1}{8} \\
&= \frac{7}{8}
\end{aligned}$$

Therefore, the correct response is **D**.

Question 2

$$P(X > 5) = a \Rightarrow P(X \le 5) = 1 - a, P(X > 8) = b \Rightarrow P(X \le 8) = 1 - b$$

$$P(X \le 5 \mid X \le 8) = \frac{P(X \le 5)}{\Pr(X \le 8)} = \frac{1-a}{1-b} = \frac{-1(a-1)}{-1(b-1)} = \frac{a-1}{b-1}$$

Therefore, the correct response is **E**.

Question 3

Write the integral for the formula $E(X)=\int_{-\infty}^{\infty} x f(x)\,dx$ and simplify.

$$\begin{aligned}\mathrm{E}(X)&=\int_0^{12}\frac{x}{288}\left(12x-x^2\right)dx\\&=\int_0^{12}\left(\frac{12x^2}{288}-\frac{x^3}{288}\right)dx\end{aligned}$$

Evaluate the integral.

$$\begin{aligned}\mathrm{E}(X)&=\left[\frac{12x^3}{288\times 3}-\frac{x^4}{288\times 4}\right]_0^{12}\\&=\left(\frac{12(12)^3}{288\times 3}-\frac{(12)^4}{288\times 4}\right)-0\\&=\mathbf{6}\end{aligned}$$

Question 4

Write the two integrals for the formula $E(X)=\int_{-\infty}^{\infty} x f(x)\,dx$ and simplify.

$$\begin{aligned}\mathrm{E}(X)&=\int_0^{2} x\times\frac{3x}{8}\,dx+\int_2^{\frac{8}{3}} x\times\left(3-\frac{9x}{8}\right)dx\\&=\int_0^{2}\frac{3x^2}{8}\,dx+\int_2^{\frac{8}{3}}\left(3x-\frac{9x^2}{8}\right)dx\end{aligned}$$

Evaluate the integrals.

$$\begin{aligned}\mathrm{E}(X)&=\left[\frac{3x^3}{3\times 8}\right]_0^{2}+\left[\frac{3x^2}{2}-\frac{9x^3}{3\times 8}\right]_2^{\frac{8}{3}}\\&=\left(\frac{3\times 2^3}{3\times 8}\right)-0+\left(\frac{3\times\left(\frac{8}{3}\right)^2}{2}-\frac{9\times\left(\frac{8}{3}\right)^3}{3\times 8}-\frac{3\times 2^2}{2}+\frac{9\times 2^3}{3\times 8}\right)\\&=1+\frac{32}{3}-\frac{64}{9}-6+3\\&=\mathbf{\frac{14}{9}}\end{aligned}$$

Question 5

(✓ = 1 mark)

a The area under the curve is 1.

$$\int_0^1 k\sin(\pi x)\,dx = 1$$

$$\left[-\frac{k}{\pi}\cos(\pi x)\right]_0^1 = 1$$

$$-\frac{k}{\pi}\cos(\pi) + \frac{k}{\pi}\cos(0) = 1$$

$$\frac{k}{\pi} + \frac{k}{\pi} = 1$$

$$\frac{2k}{\pi} = 1$$

$$k = \frac{\pi}{2}$$

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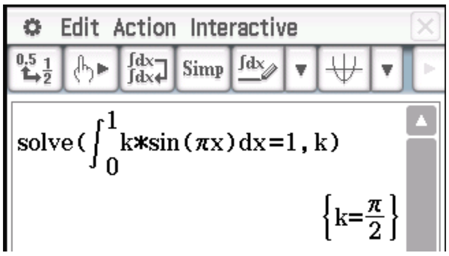

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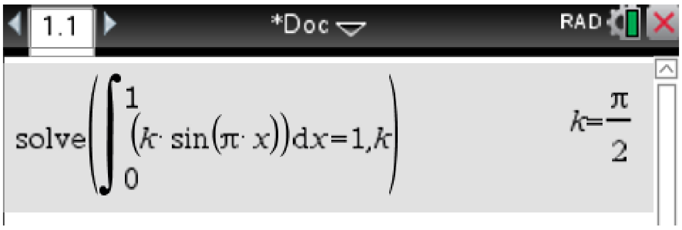
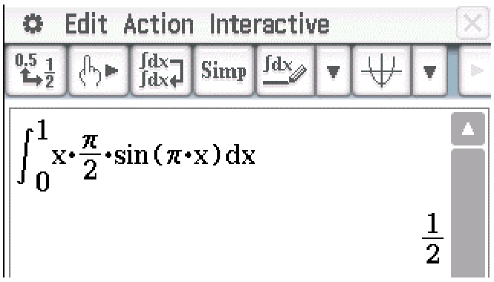


b **ClassPad**



Establish and evaluate the definite integral for the expected value using $\mathrm{E}(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$.

This gives the expected value of $\frac{1}{2}$.

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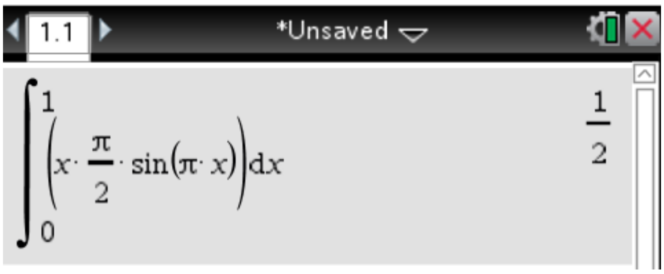

Establish and evaluate the definite integral for the expected value using $\mathrm{E}(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$.

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Question 6

The medium, m, is the x-value for which the area under the curve is 0.5.

$$\int_1^m \frac{x}{12}dx = \frac{1}{2}$$

$$\left[\frac{x^2}{24}\right]_1^m = \frac{1}{2}$$

$$\frac{m^2}{24} - \frac{1}{24} = \frac{1}{2}$$

$$m^2 = 13$$

$\boldsymbol{m = \sqrt{13}}$ or **3.61,** take positive value, since $m > 1$

Question 7

$$\int_0^k \frac{x}{8}dx = 0.3$$

$$\left[\frac{x^2}{16}\right]_0^k = 0.3$$

$$\frac{k^2}{16} = 0.3$$

$$k^2 = 4.8$$

$$k = \pm\sqrt{4.8} = \pm\sqrt{\frac{48}{10}} = \pm\sqrt{\frac{\cancel{2}\times 2\times 4\times 3}{5\times \cancel{2}}} = \pm 2\sqrt{\frac{6}{5}} = \pm\frac{2\sqrt{6\times 5}}{\sqrt{5\times 5}} = \pm\frac{2\sqrt{30}}{5}$$

$\boldsymbol{k = \frac{2\sqrt{30}}{5}}$ or **2.19,** take positive value, since $k > 0$

Question 8

$$\mathrm{Var}(X)=\mathrm{E}(X^2)-\mathrm{E}(X)^2=\int_0^1 x^2(2x)\,dx-\left(\int_0^1 x(2x)\,dx\right)^2$$

$$=\left[\frac{1}{2}x^4\right]_0^1-\left(\left[\frac{2}{3}x^3\right]_0^1\right)^2$$

$$=\frac{1}{2}-\left(\frac{2}{3}\right)^2$$

$$=\frac{1}{2}-\frac{4}{9}$$

$$=\frac{1}{18}$$

$$\mathbf{Var}(X)=\frac{\mathbf{1}}{\mathbf{18}}=\mathbf{0.0\dot{5}}$$

$$\mathbf{SD}(X)=\sqrt{\frac{1}{18}}=\frac{1}{18}\sqrt{18}=\frac{1}{6}\sqrt{2}=\frac{\sqrt{2}}{6}=\mathbf{0.2357}$$

Question 9

a $$\mathrm{E}(Y)=\mathrm{E}\left(\frac{2}{3}X-1\right)=\frac{2}{3}\mathrm{E}(X)-\mathrm{E}(1)$$

$$=\frac{2}{3}\times 3-1$$

$$\mathbf{=1}$$

b $$\mathrm{Var}(Y)=\mathrm{Var}\left(\frac{2}{3}X-1\right)$$

$$=\left(\frac{2}{3}\right)^2\mathrm{Var}(X)$$

$$=\left(\frac{2}{3}\right)^2\times 9$$

$$\mathbf{=4}$$

c $\mathrm{SD}(Y)=\sqrt{\mathrm{Var}(Y)}=\sqrt{4}=\mathbf{2}$

Question 10 (6 marks)

(✓ = 1 mark)

a $\mathbf{Var}(X) = \mathbf{E}(X^2) - \mathbf{E}(X)^2$ ✓

so $\mathrm{E}(X^2) = \mathrm{Var}(X) + \mathrm{E}(X)^2$

$= 5 + 2^2$

$= \mathbf{9}$ ✓

b $\mathbf{E}(3X + 2) = 3\mathbf{E}(X) + \mathbf{E}(2)$ ✓

$= 3 \times 2 + 2$

$= \mathbf{8}$ ✓

c $\mathbf{Var}\left(-\frac{1}{5}X - 1\right) = \left(\frac{1}{5}\right)^2 \mathbf{Var}(X)$ ✓

$= \frac{1}{25} \times 5$

$= \frac{\mathbf{1}}{\mathbf{5}}$ ✓

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Question 11 (4 marks)

(✓ = 1 mark)

a

$$f(x)=\begin{cases}\dfrac{k}{x^2} & 1\le x\le 2\\ 0 & \text{elsewhere}\end{cases}$$

$$\int_1^2 \frac{k}{x^2}\,dx=1$$

$$\int_1^2 kx^{-2}\,dx=1$$

$$\left[-kx^{-1}\right]_1^2=1$$

$$\left[-\frac{k}{x}\right]_1^2=1$$

$$-\frac{k}{2}-\left(-\frac{k}{1}\right)=1$$

$$-\frac{k}{2}+k=1$$

$$\frac{k}{2}=1$$

$$\boldsymbol{k=2}$$ ✓

b

From part **a**, $k=2$ so the probability function is $p(x)=\dfrac{2}{x^2}$

$$\mathrm{E}(X)=\mu=\sum xp(x)\,dx$$

$$\mathrm{E}(X)=\int_1^2 x\frac{2}{x^2}\,dx$$

$$=\boldsymbol{\int_1^2 \frac{2}{x}\,dx}$$ ✓

$$=\left[2\log_e(x)\right]_1^2$$

$$=2\log_e(2)-2\log_e(1)$$

$$=2\log_e(2)-0$$

$$=\mathbf{2\ln(2)}$$ ✓

Question 12 (3 marks)

(✓ = 1 mark)

Using the definition of the median m,

$$\int_0^m \left(\frac{1}{5}e^{-\frac{x}{5}}\right)dx = \frac{1}{2}$$

$$\left[-e^{-\frac{x}{5}}\right]_0^m = \frac{1}{2}$$

$$-e^{-\frac{m}{5}} - (-1) = \frac{1}{2} \checkmark$$

$$1 - e^{-\frac{m}{5}} = \frac{1}{2}$$

$$e^{-\frac{m}{5}} = \frac{1}{2} \text{ or } e^{\frac{m}{5}} = 2$$

$$m = 5\ln(2) \text{ or } m = -5\ln\left(\frac{1}{2}\right) \text{ or } m = \ln(32) \checkmark$$

Question 13 (7 marks)

(✓ = 1 mark)

a $\int_0^5 ax(5-x)\,dx = 1 \checkmark$

$$a\left[x^2\left(\frac{5}{2} - \frac{1}{3}x\right)\right]_0^5 = 1 \checkmark$$

$$25a\left(\frac{5}{2} - \frac{5}{3}\right) = 1$$

$$\frac{125a}{6} = 1$$

$$a = \frac{6}{125} \checkmark$$

b Given that the pdf is a parabola with roots at $x = 0$ and $x = 5$, the line of symmetry has the equation $x = \frac{5}{2}$. So, $f(x)$ is symmetrical about $x = \frac{5}{2}$ over $0 \le x \le 5$ and hence $\mathrm{E}(x) = \frac{5}{2}$.

c $$\text{Var}(X) = \text{E}(X^2) - \text{E}(X)^2$$

$$= \frac{6}{125}\int_0^5 (5x^3 - x^4)\,dx - \left(\frac{5}{2}\right)^2 \checkmark$$

$$= \frac{6}{125}\left[\frac{5x^4}{4} - \frac{x^5}{5}\right]_0^5 - \frac{25}{4}$$

$$= \frac{6}{125}\left(\frac{5^5}{4} - \frac{5^5}{5}\right) - \frac{25}{4}$$

$$= \frac{6}{125}\left(\frac{5^5}{20}\right) - \frac{25}{4}$$

$$= \frac{6 \times 25}{20} - \frac{5 \times 25}{20}$$

$$= \frac{25}{20}$$

$$= \frac{5}{4} \checkmark$$

$$\text{SD}(X) = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2} \checkmark$$

Question 14 [SCSA MM2018 Q10c] (2 marks)

(✓ = 1 mark)

Solve $\int_0^m f(x)\,dx = 0.5$ over domain $0 \le x \le 2$

$$\int_0^m f(x)\,dx = -\frac{3m^4}{16} + \frac{m^3}{2}$$

for median: $-\frac{3m^4}{16} + \frac{m^3}{2} = 0.5$

$m = 1.23$

uses correct integral✓

determines m = 1.23✓

Question 15 **[SCSA MM2017 Q11bc] (7 marks)**

(✓ = 1 mark)

a

$$E(X)=\int_0^1 x\left(\frac{4}{3}-\frac{2}{3}x\right)dx=\frac{2}{3}x^2-\frac{2}{9}x^3\Big|_0^1=\frac{2}{3}-\frac{2}{9}=\frac{4}{9}\approx 0.4444$$

That is, 27 minutes.

writes the correct integral for the mean✓

calculates the mean correctly✓

converts to minutes✓

b

$$\text{Var}(X)=\int_0^1\left(x-\frac{4}{9}\right)^2\left(\frac{4}{3}-\frac{2}{3}x\right)dx$$
$$=0.0802$$

or

$$E\left(X^2\right)=\int_0^1 x^2\left(\frac{4}{3}-\frac{2}{3}x\right)dx=\frac{4}{9}x^3-\frac{1}{6}x^4\Big|_0^1=\frac{4}{9}-\frac{1}{6}=\frac{5}{18}\approx 0.2778$$

So $\text{Var}(X)=\frac{5}{18}-\frac{16}{81}=\frac{13}{162}\approx 0.0802$

$\sigma=\sqrt{0.0802}\approx 0.2833$

That is, 17 minutes.

calculates $E(X^2)$ correctly or states integral for VAR✓

calculates the variance correctly✓

calculates standard deviation correctly✓

converts to minutes✓

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Question 16 (10 marks)

(✓ = 1 mark)

a

$$\lim_{k\to\infty}\left(\int_0^k \frac{1}{8}e^{-\frac{x}{8}}dx\right) ✓$$

$$= \lim_{k\to\infty}\left(\left[-e^{-\frac{x}{8}}\right]_0^k\right)$$

$$= \lim_{k\to\infty}\left(-e^{-\frac{k}{8}}+1\right)$$

$$= 0+1 = 1 ✓$$

All values of $f(x)$ are positive and the sum of all probabilities equals 1. Therefore, it is a valid pdf.✓

b **i**

$$E(X) = \int_0^\infty \frac{1}{8}xe^{-\frac{x}{8}}dx ✓$$

$$= 8 ✓$$

ii

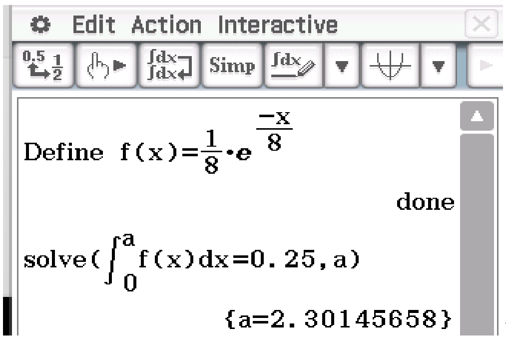
$$\text{Var}(X) = E\left(X^2\right) - E(X)^2 ✓$$

$$= \int_0^\infty \frac{1}{8}x^2e^{-\frac{x}{8}}dx - 8^2 ✓$$

$$= 128 - 64 ✓$$

$$= 64$$

c **ClassPad**



✓

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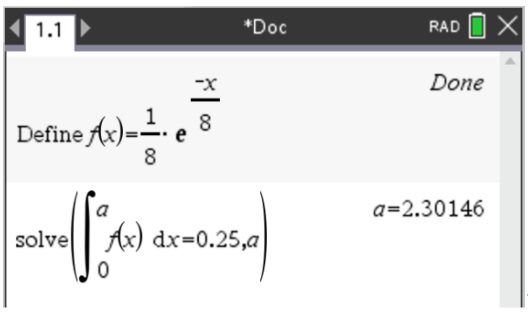


✓

The value of k is 2.301✓

Question 17 (10 marks)

(✓ = 1 mark)

a

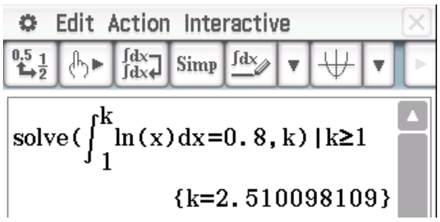
$$\mathbf{P}(X>1\,|\,X<2)=\frac{\mathbf{P}(1<X<2)}{\mathbf{P}(X<2)}\checkmark$$

$$=\frac{\mathrm{P}(1<X<2)}{\mathrm{P}(1<X<2)}=\mathbf{1}\checkmark$$

b

Solve for k: $\int_1^k \log_e(x)\,dx = 0.8$ ✓

ClassPad



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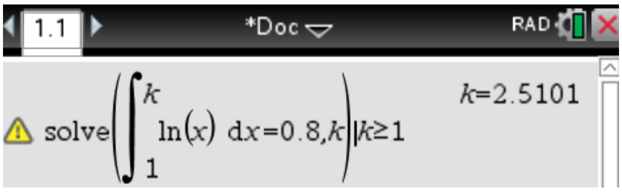


$\boldsymbol{k = 2.5101}$ ✓

c

$$\mathrm{E}(X)=\int_{-\infty}^{\infty} x f(x)\,dx=\int_1^e x\log_e(x)\,dx\ \checkmark$$

Using CAS gives **E(*X*) = 2.0973**✓

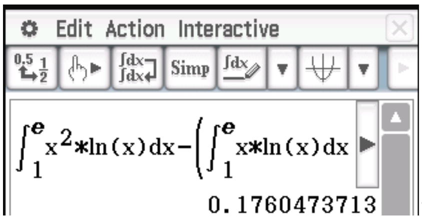
d

$$\mathrm{Var}(X)=\int_1^e x^2\ln(x)\,dx-\mathrm{E}(X)^2\ \checkmark$$

$$=\mathbf{0.1760}\ \checkmark$$

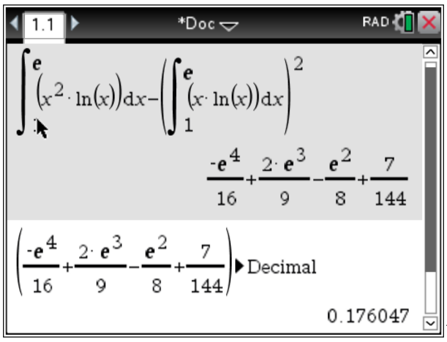
or using CAS:

ClassPad



✓

TI-Nspire



✓

Var(*X*) = 0.1760✓

e

$$\mathbf{SD}(-2X)=|-2|\times\mathbf{SD}(X)\checkmark$$

$$=2\times\sqrt{0.1760}$$

$$=0.8390$$

SD(*Y*) = 0.8390✓

Question 18 (5 marks)

(✓ = 1 mark)

a To determine the expected value of *X*, write the integral for the formula.

$$\mathbf{E}(X) = \int_{-\infty}^{\infty} xg(x)\,dx ✓$$

$$= \int_1^3 \left(x \times \frac{(x-3)^2 + 64}{256} \right) dx + \int_3^5 \left(x \times \frac{x+29}{128} \right) dx$$

$$= \frac{37}{12}$$

$$\approx \mathbf{3.0833} ✓$$

b Use the pdf to determine the probability that a randomly chosen member takes more than 4 minutes to complete *S*.

$$\mathbf{P}(X > 4) = \int_4^5 \left(\frac{x+29}{128} \right) dx ✓$$

Multiplying this by 200 gives the number of members expected to meet this condition.

$$\int_4^5 \left(\frac{x+29}{128} \right) dx \times 200$$

$$= \mathbf{52.34} ✓$$

So, **52**✓ members are expected to take more than 4 minutes to complete *S*.

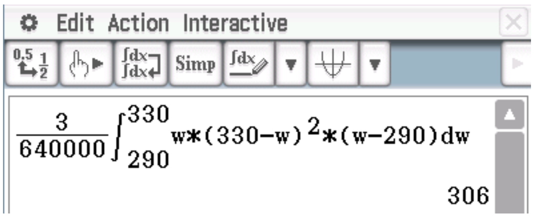
Question 19 (8 marks)

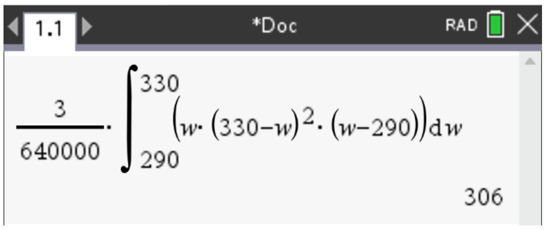
(✓ = 1 mark)

a The mean weight is the expected value of W. To calculate this, write the integral for the formula

$$\mathbf{E}(W)=\int_{-\infty}^{\infty} w\ C(w)dw=\frac{3}{640\ 000}\int_{290}^{330} w(330-w)^2(w-290)\,dw\ ✓$$

$$= \mathbf{306\ g}✓$$

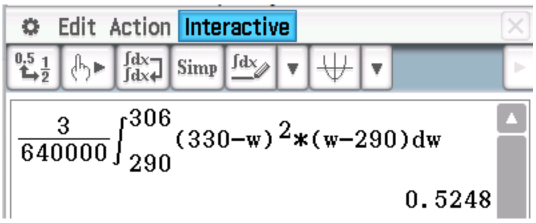


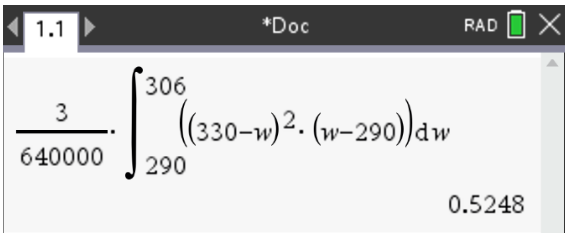


b To find the probability that a randomly selected controller weighs less than the mean weight of the controllers, that is,

$$\mathbf{P}(W)=\frac{3}{640\ 000}\int_{290}^{306}(330-w)^2(w-290)\,dw\ ✓$$

$$= \mathbf{0.5248}\ ✓$$





c $\mathrm{E}(W)=306$

$$\mathrm{E}(W^2)=\frac{3}{640\ 000}\int_{290}^{330} w^2(330-w)^2(w-290)\,dw=93\ 700$$

$$\mathbf{Var}(W)=\mathbf{E}(W^2)-\mathbf{E}(W)^2=\mathbf{93700-306^2=64}\ ✓$$

$$\mathrm{SD}(W)=\sqrt{64}=\mathbf{8\ g}✓$$

d More than 1 standard deviation from mean is P(W > 306 + 8) = **P(W > 314)**✓

$$\mathrm{P}(W>314)=\frac{3}{640\ 000}\int_{314}^{330}(330-w)^2(w-290)\,dw$$

$$= \mathbf{0.1792}✓$$

EXERCISE 8.3 Uniform and triangular distribution

Question 1

Use CAS to evaluate $\int_0^{\infty} 0.05te^{-0.05t}dt$ to get 20.

Hence the answer is **E**.

Question 2

The 50th percentile is found using $\int_0^{0.5} \frac{\pi}{2}\sin(\pi x)dx$, which is calculated to be 0.5

Hence the answer is **A**.

Question 3

a $\mathbf{P(X=10)=0}$, by definition.

b

$$\begin{aligned} P(X>10) &= \int_{10}^{15} \frac{1}{6}\,dx \\ &= \left[\frac{x}{6}\right]_{10}^{15} \\ &= \frac{1}{6}(15-10) \\ &= \mathbf{\frac{5}{6}} \end{aligned}$$

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c $P(X \le 12 \mid X > 10) = \dfrac{P(10 < X \le 12)}{P(X > 10)}$

$P(X > 10) = \dfrac{5}{6}$, from part **b**.

$$P(10 < X \le 12) = \int_{10}^{12} \frac{1}{6}\, dx$$

$$= \left[\frac{x}{6}\right]_{10}^{12} = \frac{1}{3}$$

Hence $P(X \le 12 \mid X > 10) = \dfrac{\frac{1}{3}}{\frac{5}{6}} = \dfrac{6}{15} = \mathbf{\dfrac{2}{5}}$

Question 4

a

Compare $F(t) = \begin{cases} 0 & t < 7 \\ \dfrac{t-7}{13} & 7 \le t \le b \\ 1 & t > b \end{cases}$ to $F(t) = \begin{cases} 0 & t < a \\ \dfrac{t-a}{b-a} & a \le t \le b \\ 1 & t > b \end{cases}$

Hence, $a = 7$ and $b - a = 13$, so $\boldsymbol{b = 20}$

b

Use $f(x) = \dfrac{1}{b-a}$ or $f(x) = \dfrac{d}{dx}(F(x))$ to have $f(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$

$$\boldsymbol{f(x) = \begin{cases} \dfrac{1}{13} & 7 \le x \le 20 \\ 0 & \textbf{otherwise} \end{cases}}$$

c $P(T > 15 \mid T > 10) = \dfrac{P(T > 15)}{P(T > 10)}$

$$P(T > 15) = \int_{15}^{20} \frac{1}{13}\, dx = \left[\frac{x}{13}\right]_{15}^{20} = \frac{5}{13}$$

$$P(T > 10) = \int_{10}^{20} \frac{1}{13}\, dx = \left[\frac{x}{13}\right]_{10}^{20} = \frac{10}{13}$$

$$P(T > 15 \mid T > 10) = \frac{\frac{5}{13}}{\frac{10}{13}} = \mathbf{\frac{1}{2}}$$

Question 5

a

Identify the values of a and b in $f(x)=\begin{cases}\dfrac{1}{b-a} & a\le x\le b\\ 0 & \text{otherwise}\end{cases}$

$X\sim U[-2,48]$, so $a=-2$, $b=48$

Hence, $\boldsymbol{f(x)=\begin{cases}\dfrac{1}{50} & -2\le x\le 48\\ 0 & \text{otherwise}\end{cases}}$

b

$$\mathrm{E}(X)=\frac{a+b}{2}=\frac{-2+48}{2}=\mathbf{23}$$

c

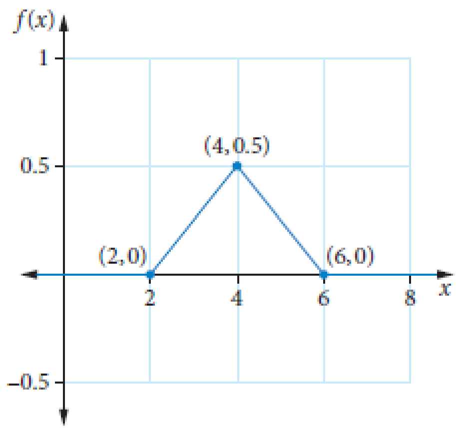
$$\mathrm{Var}(X)=\frac{(b-a)^2}{12}=\frac{(48+2)^2}{12}=\mathbf{\frac{625}{3}}=\mathbf{208.3}$$

d

$$\mathrm{SD}(X)=\sqrt{\mathrm{Var}(X)}=\sqrt{\frac{625}{3}}=\mathbf{\frac{25\sqrt{3}}{3}}=\mathbf{14.43}$$

Question 6

a



Line $f(x)=\frac{1}{4}x-\frac{1}{2}$

$x=2,\ f(2)=0 \Rightarrow (2, 0)$

$x=4,\ f(4)=1-\frac{1}{2}=0.5 \Rightarrow (4, 0.5)$

Line $f(x)=\frac{3}{2}-\frac{1}{4}x$

$x=4,\ f(4)=\frac{3}{2}-1=\frac{1}{2} \Rightarrow (4, 0.5)$

$x=6,\ f(6)=\frac{3}{2}-\frac{3}{2}=0 \Rightarrow (6, 0)$

It is a symmetrical triangular distribution about $x = 4$.

b i

$$\mathrm{P}(X<3)=\int_2^3\left(\frac{1}{4}x-\frac{1}{2}\right)dx=\left[\frac{1}{8}x^2-\frac{1}{2}x\right]_2^3$$
$$=\left(\frac{9}{8}-\frac{3}{2}\right)-\left(\frac{1}{2}-1\right)=\mathbf{\frac{1}{8}}$$

Or find the area of the triangle with base $(3 - 2 = 1)$ and height $\frac{1}{4}\times 3-\frac{1}{2}=\frac{1}{4}$

Area is $\frac{1}{2}\times 1\times\frac{1}{4}=\mathbf{\frac{1}{8}}$

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ii $$\text{P}(X<3\mid X<4)=\frac{\text{P}(X<3)}{\text{P}(X<4)}$$

$\text{P}(X<3)=\frac{1}{8}$, from part **b i** and $\text{P}(X<4)=\frac{1}{2}$, from the symmetry of the triangle, where $x=4$ is the midpoint of the base of the triangle.

Hence $\text{P}(X<3\mid X<4)=\frac{\frac{1}{8}}{\frac{1}{2}}=\mathbf{\frac{1}{4}}$

iii Use $\text{E}(X)=\frac{a+b}{2}$ for each line and then find the mean.

For $2\le x\le 4$, expected value is $\frac{4+2}{2}=3$

For $4\le x\le 6$, expected value is $\frac{4+6}{2}=5$

So $\mathbf{E(X)=\frac{3+5}{2}=4}$

iv Use $\text{Var}(X)=\frac{(b-a)^2}{12}$

For $2\le x\le 4$, variance is $\frac{(4-2)^2}{12}=\frac{1}{3}$

For $4\le x\le 6$, variance is $\frac{(6-4)^2}{12}=\frac{1}{3}$

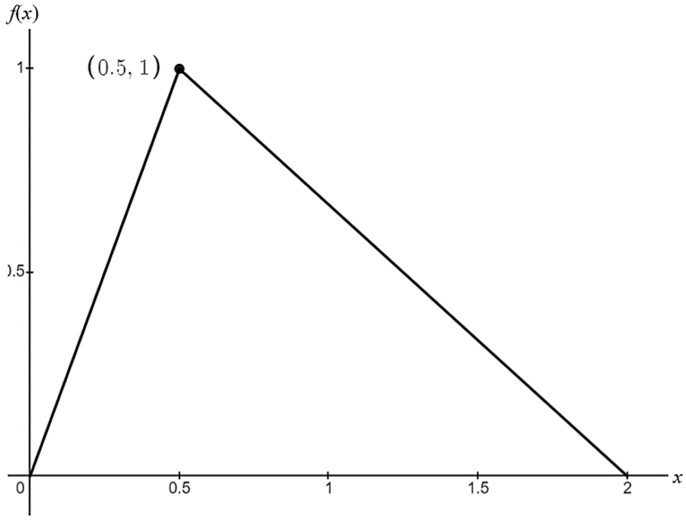
So $\mathbf{Var(X)=\frac{1}{3}+\frac{1}{3}=\frac{2}{3}}$

Question 7

a

Show that $f(x) \geq 0$ for all x in $a \leq x \leq b$ and $\int_a^b f(x)\,dx = 1$.

Draw the graph.



Hence $\boldsymbol{f(x) \geq 0}$ **for all** $\boldsymbol{x}$ **in** $\boldsymbol{0 \leq x \leq 2}$

$\int_0^2 f(x)\,dx$ is the area of the triangle.

Area is $\boldsymbol{\frac{1}{2} \times 2 \times 1 = 1}$, hence $\boldsymbol{\int_0^2 f(x)\,dx = 1}$

or

$$\int_0^{\frac{1}{2}} 2x\,dx + \int_{\frac{1}{2}}^2 -\frac{2}{3}x + \frac{4}{3}dx$$

$$= \left[x^2\right]_0^{\frac{1}{2}} + \left[-\frac{x^2}{3} + \frac{4x}{3}\right]_{\frac{1}{2}}^2$$

$$= \frac{1}{4} + \left(-\frac{4}{3} + \frac{8}{3} + \frac{1}{12} - \frac{2}{3}\right)$$

$$= \frac{3 + 16 + 1 - 8}{12}$$

$$= 1$$

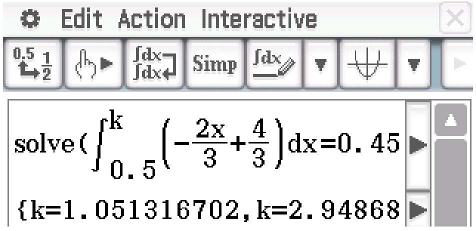
All values of $f(x)$ are positive and the sum of all probabilities equals 1. Therefore, it is a valid probability density function.

b The 70th percentile is the value of $x = k$ so that the area under the graph is 0.7.

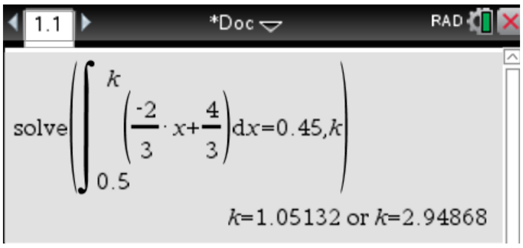
The area of the triangle from $x = 0$ to $x = 0.5$ is 0.25, so k lies in the interval [1, 2] and the area under the graph is 0.7 – 0.25 = 0.45.

We want the value of k so that $\int_{0.5}^{k}\left(-\frac{2}{3}+\frac{4}{3}x\right)dx = 0.45$.

ClassPad



TI-Nspire



Reject 2.948 since $k < 2$.

The 70th percentile is 1.05

c

$$\mathrm{E}(X) = \int_0^{\frac{1}{2}} x \times 2x\,dx + \int_{\frac{1}{2}}^{2} x\left(-\frac{2}{3}x+\frac{4}{3}\right)dx$$

$$= \int_0^{\frac{1}{2}} 2x^2 dx + \int_{\frac{1}{2}}^{2}\left(-\frac{2}{3}x^2+\frac{4}{3}x\right)dx$$

$$= \left[\frac{2}{3}x^3\right]_0^{\frac{1}{2}} + \left[-\frac{2}{9}x^3+\frac{2}{3}x^2\right]_{\frac{1}{2}}^{2}$$

$$= \frac{2}{3}\times\left(\frac{1}{2}\right)^3 + \left(-\frac{16}{9}+\frac{8}{3}\right) - \left(-\frac{2}{9}\times\left(\frac{1}{2}\right)^3+\frac{2}{3}\times\left(\frac{1}{2}\right)^2\right)$$

$$= \frac{1}{12}+\frac{8}{9}-\left(-\frac{1}{36}+\frac{1}{6}\right)$$

$$= \frac{1}{12}+\frac{8}{9}-\frac{5}{36}$$

$$= \frac{3+32-5}{36}$$

$$= \frac{30}{36}$$

$$= \frac{5}{6}$$

Question 8

a Use $\mathrm{E}(X)=\dfrac{a+b}{2}$ and $\mathrm{SD}(X)=\dfrac{b-a}{\sqrt{12}}$

$$\mathrm{E}(L)=\frac{245+251}{2}=248$$

The mean of *L* is 248 mm

$$\mathrm{SD}(L)=\frac{251-245}{\sqrt{12}}=\sqrt{3}\approx 1.73$$

The standard deviation of *L* is 1.73 mm

b Determine the value of $f(x)$ using $\dfrac{1}{b-a}$.

$$f(x)=\frac{1}{251-245}=\frac{1}{6}$$

$$\mathrm{P}(L>250)=\frac{1}{6}\times(251-250)=\mathbf{\frac{1}{6}}$$

c Let X be the number of pipes out of 60 whose length is greater than 250 mm.

Determine the value of $f(x)$ using $X\sim \mathrm{Bin}\left(60,\dfrac{1}{6}\right)$.

$$\mathrm{E}(X)=np=60\times\frac{1}{6}=10$$

10 pipes are expected to have a length greater than 250 mm.

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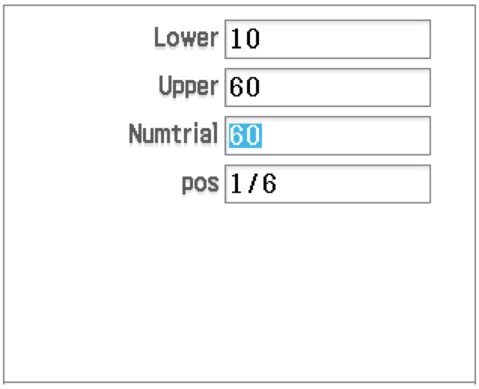
d Let X be the number of pipes out of 60 whose length is greater than 250 mm.

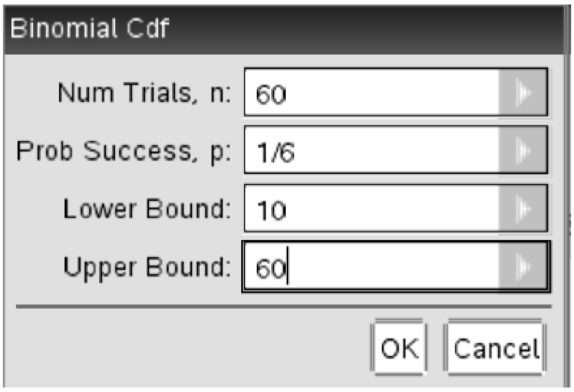
Then determine the value of $f(x)$ using $X \sim \text{Bin}\left(60, \frac{1}{6}\right)$.

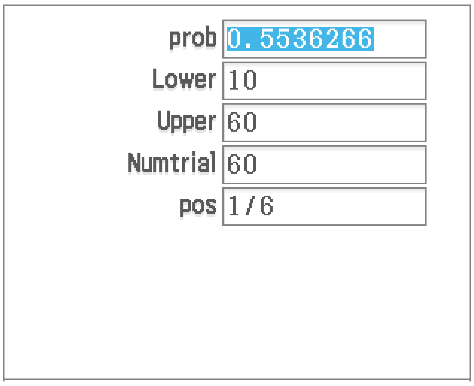
Use CAS to find $\text{P}(X \geq 10)$.

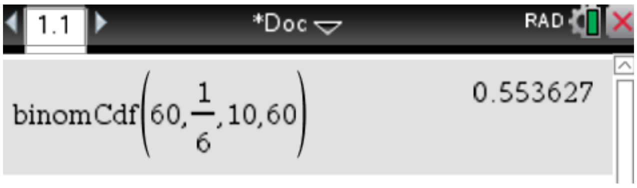
ClassPad



TI-Nspire







$\mathbf{P(X \geq 10) = 0.5536}$

Question 9 [SCSA MM2019 Q3 MODIFIED] (8 marks)

(✓ = 1 mark)

a For $0 \le t \le 1.5$ the gradient of the line is $\frac{0.5}{1.5} = \frac{1}{3}$. The line passes through the origin, so the equation is $f(t) = \frac{1}{3}t$

For $1.5 \le t \le 4$ the gradient of the line is $-\frac{0.5}{2.5} = -\frac{1}{5}$.

So $f(t) = -\frac{1}{5}t + c$

The point (4, 0) lies on the line, so $0 = -\frac{1}{5} \times 4 + c \Rightarrow c = \frac{4}{5}$. Hence $f(t) = -\frac{1}{5}t + \frac{4}{5}$

Therefore $f(t) = \begin{cases} \frac{t}{3} & 0 \le t \le 1.5 \\ -\frac{t}{5} + \frac{4}{5} & 1.5 \le t \le 4 \\ 0 & \text{otherwise} \end{cases}$

calculates the two gradients correctly✓

obtains the correct equation for each interval✓

displays the results in the correct format✓

b $P(T \le 1) = \int_0^1 \frac{t}{3} dt$

$= \left[\frac{t^2}{6}\right]_0^1$

$= \frac{1}{6}$

choose the correct equation for $f(t)$ from part a when $0 \le t \le 1.5$✓

evaluates integral to determine probability✓

or

Required probability is the area of the triangle that has base 1 unit.

The height of the triangle is $f(1) = \frac{1}{3}$, from part **a**.

$P(T \le 1) = \frac{1}{2} \times 1 \times \frac{1}{3} = \frac{1}{6}$

determines the height of the triangle✓

correctly calculates the area✓

c $P(1 \le T \le 3) = 1 - P(0 \le T \le 1) - P(3 \le T \le 4)$

$P(0 \le T \le 1) = \frac{1}{6}$

$P(3 \le T \le 4) = \frac{1}{5}\int_3^4 (-t+4)\, dt = \frac{1}{5}\left[-\frac{t^2}{2} + 4t\right]_3^4$

$= \frac{1}{5}\left(-8 + 16 + \frac{9}{2} - 12\right)$

$= \frac{1}{10}$

$P(1 \le T \le 3) = 1 - \frac{1}{6} - \frac{1}{10} = \frac{22}{30} = \frac{11}{15}$

choose the correct equation for $f(t)$ from part a when $1.5 \le t \le 4$✓

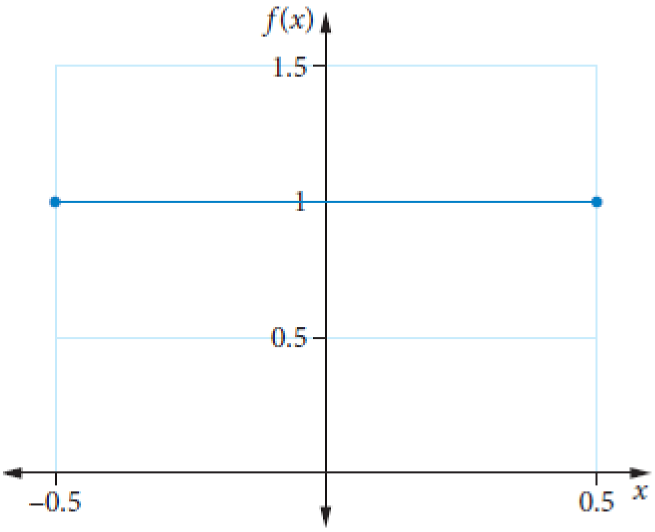
calculates $P(3 \le t \le 4)$ correctly✓

calculates $P(1 \le t \le 3)$ correctly✓

Question 10 [SCSA MM2019 Q6 MODIFIED] (11 marks)

(✓ = 1 mark)

a



This is a continuous uniform distribution.

draws horizontal line at $y = 1$✓

indicates a domain of $-0.5 < x < 0.5$✓

states that the graph describes a uniform distribution✓

b $\mathrm{P}(X \geq 0.35) = \text{Area} = 0.15$

computes the correct probability✓

c $\mathrm{P}(X < -0.35 \mid X < 0) = \dfrac{\mathrm{P}(X < -0.35 \cap X < 0)}{\mathrm{P}(X < 0)} = \dfrac{\mathrm{P}(X < -0.35)}{\mathrm{P}(X < 0)} = \dfrac{0.15}{0.5} = 0.3$

writes the correct conditional probability statement✓

computes the probability correctly✓

d $\mathrm{P}(X^2 < 0.09) = \mathrm{P}(-0.3 < X < 0.3) = 0.6$

correctly expresses the required probability in terms of X✓

computes the probability correctly✓

e $E(X) = \int_{-0.5}^{0.5} x(1)\,dx = 0$

So $\mathrm{Var}(X) = \int_{-0.5}^{0.5} (x-0)^2 (1)\,dx = \left.\dfrac{x^3}{3}\right|_{-0.5}^{0.5} = \dfrac{0.125 + 0.125}{3} = \dfrac{1}{12}$

computes mean correctly✓

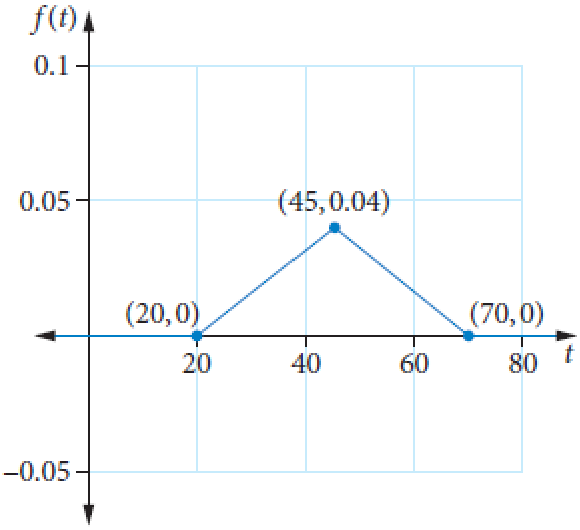
states an integral for the variance✓

evaluates the integral to determine variance correctly✓

Question 11 (10 marks)

(✓ = 1 mark)

a



correctly drawn graph with straight lines✓

horizontal lines on the *t*-axis for the domains $0 \le t < 20$ and $t > 70$✓

two diagonal lines for the domain $20 \le t < 70$✓

b **The shape describes a symmetrical triangular distribution about $t = 45$.✓**

c **i** For ClassPad users, define the separate parts of the piecewise function.

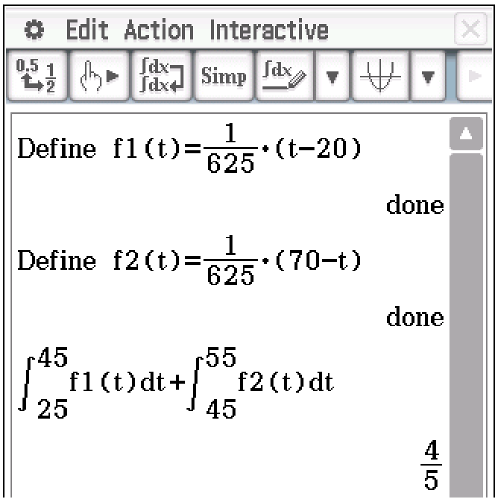
$f_1(t)=\frac{1}{625}(t-20)$ and $f_2(t)=\frac{1}{625}(70-t)$

Calculate $\mathbf{P(25 \le T \le 55)=\int_{25}^{45} f_1(t)\,dt+\int_{45}^{55} f_2(t)\,dt}$.✓

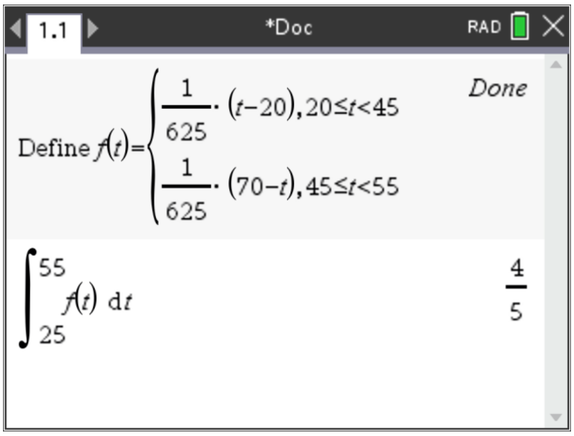
For TI-Nspire users, define the piecewise function $f(t)=\begin{cases}\frac{1}{625}(t-20) & 20 \le t < 45\\ \frac{1}{625}(70-t) & 45 \le t \le 55\end{cases}$

Calculate $\mathrm{P}(25 \le T \le 55)=\int_{25}^{55} f(t)\,dt$.✓

ClassPad



TI-Nspire



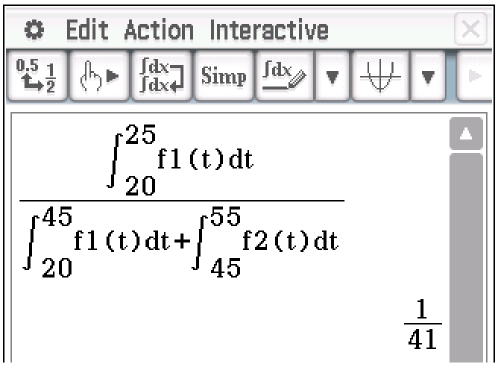
$\mathrm{P}(25 \le T \le 55)=\frac{\mathbf{4}}{\mathbf{5}}=\mathbf{0.8}$✓

ii $$\mathbf{P}(T \le 25 \mid T \le 55) = \frac{\mathbf{P}(T \le 25 \cap T \le 55)}{\mathbf{P}(T \le 55)}$$

$$= \frac{\mathbf{P}(T \le 25)}{\mathbf{P}(T \le 55)} \quad ✓$$

$$= \frac{\int_{20}^{25} f(t)\,dt}{\int_{20}^{55} f(t)\,dt}$$

ClassPad



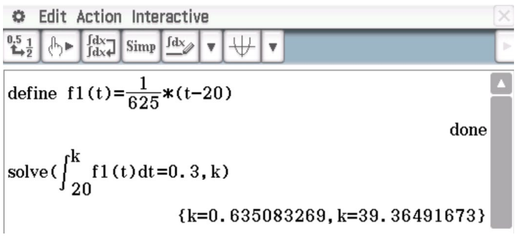
TI-Nspire

$$\mathrm{P}(T \le 25 \mid T \le 55) = \frac{\mathbf{1}}{\mathbf{41}} \text{ or } \approx \mathbf{0.02}✓$$

iii $\mathrm{P}(T \ge k) = 0.7 \Rightarrow \mathrm{P}(T \le k) = 0.3$

Solve $\int_{20}^{k} \frac{1}{625}(t-20)\,dt \le 0.3$ **for** ***k***.✓

ClassPad



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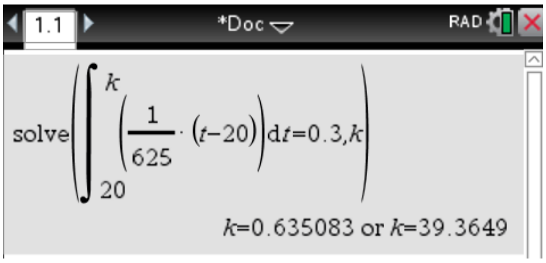


Rounding to four decimal places, $\boldsymbol{k} = \mathbf{39.3649}$.✓

Question 12 (8 marks)

(✓ = 1 mark)

a The first two parts are straight lines. We can easily sketch them by finding their end points.

For $10 \le t \le 20$:

When $t = 10, f(t) = (t - 10)/100 = 0$

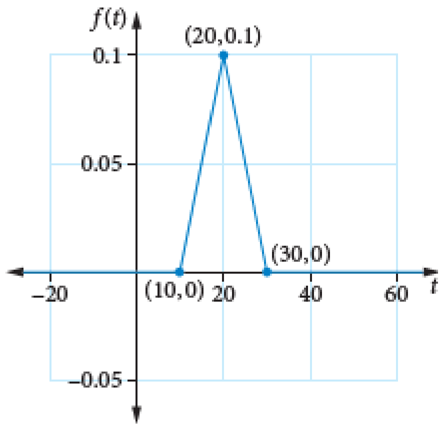
When $t = 20, f(t) = (20 - 10)/100 = 0.1$

For $20 \le t \le 30$:

When $t = 20, f(t) = (30 - 20)/100 = 0.1$

When $t = 30, f(t) = (30 - 30)/100 = 0$✓

Sketch the graph, including $f(t) = 0$ for $t < 10$ and $t > 30$.



graph is symmetrical with $t = 20$ as the axis of symmetry✓

coordinates of x-intercepts and maximum value shown✓

b **The graph is a symmetrical triangular distribution about $t = 20$.✓**

c **i**

$$P(T<25)=\frac{1}{100}\int_{10}^{20}(t-10)\,dt+\frac{1}{100}\int_{20}^{25}(30-t)\,dt=\frac{1}{100}\left(0.5+\int_{20}^{25}(30-t)\,dt\right)✓$$

Alternatively, $P(T<25)=1-\frac{1}{100}\int_{25}^{30}(30-t)\,dt$✓

Entering either of these expressions into a CAS will give $P(T<25)=\mathbf{\frac{7}{8}}$✓

ii

$$P(T\le 15\,|\,T\le 25)=\frac{P(T\le 15\cap T\le 25)}{P(T\le 25)}=\frac{\mathbf{P(T\le 15)}}{\mathbf{P(T\le 25)}}✓$$

$P(T\le 25)=\frac{7}{8}$ from part **b**.

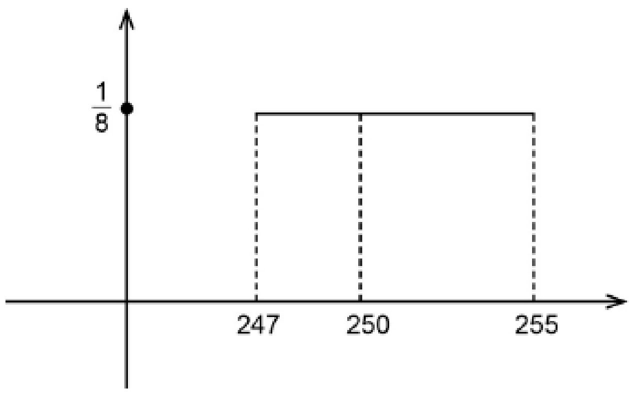
$P(T\le 15)=\frac{1}{100}\int_{10}^{15}(t-10)\,dt=\frac{1}{8}$ (using CAS)

$$P(T\le 15\,|\,T\le 25)=\frac{P(T\le 15)}{P(T\le 25)}=\frac{1}{8}\div\frac{7}{8}=\mathbf{\frac{1}{7}}✓$$

Question 13 [SCSA MM2016 Q16] (10 marks)

(✓ = 1 mark)

a



$P(X<250)=\frac{3}{8}$

evaluates pdf value✓

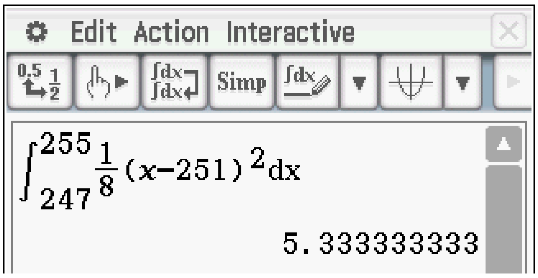
uses area to calculate probability✓

determines probability✓

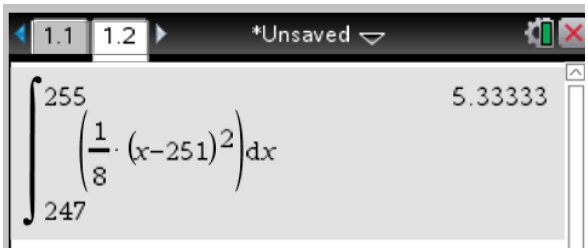
b $\text{Mean} = \dfrac{255+247}{2} = 251$

The variance can be calculated using CAS:

ClassPad



TI-Nspire



$\text{Standard deviation} = \sqrt{\text{Var}(X)} = 2.31$

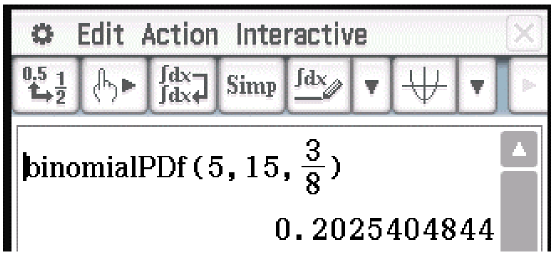
determines mean✓

uses appropriate pdf function✓

sets up an integral to find variance✓

determines standard deviation✓

c **ClassPad**



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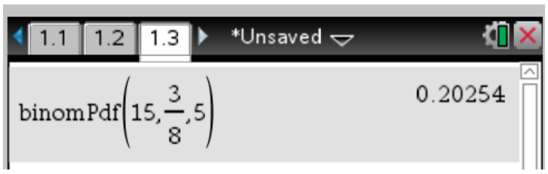


uses binomial distribution for first 15 selections✓

determines probability for the first five bottles✓

determines final probability✓

Question 14 (9 marks)

(✓ = 1 mark)

a

Use $\mathrm{E}(W)=\dfrac{a+b}{2}$ and $\mathrm{SD}(W)=\dfrac{b-a}{\sqrt{12}}$

$$\mathrm{E}(W)=\frac{245+253}{2}=249$$

The mean of *W* is 249 g✓

$$\mathrm{SD}(W)=\frac{253-245}{\sqrt{12}}\approx 2.31$$

The standard deviation of *W* is 2.31 g✓

b

Determine the value of $f(x)$ using $\dfrac{1}{b-a}$.

$$f(x)=\frac{1}{253-245}=\frac{1}{8}\ ✓$$

$$\mathrm{P}(W>250)=\frac{1}{8}\times(253-250)=\mathbf{\frac{3}{8}}\ ✓$$

c

$$\mathbf{E}(X)=np\ ✓$$

$$=100\times\frac{3}{8}=37.5\ ✓$$

Approximately 38 packets are expected to weigh more than 250 g.✓

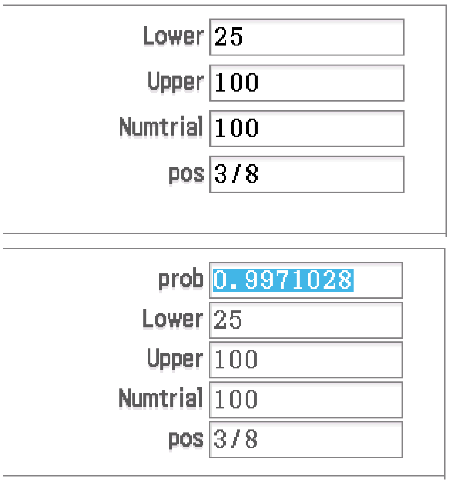
d

We use a binomial distribution.

Let X be the number of packets that weigh more than 250 g.

Then we have $X\sim \mathrm{Bin}\left(100,\dfrac{3}{8}\right)$ ✓

ClassPad



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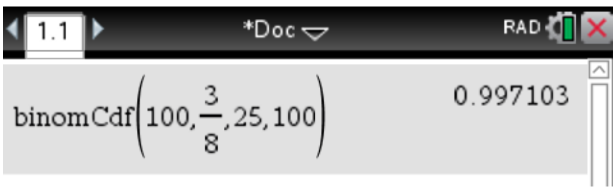

The probability that at least 25 packets weigh more than 250 g is 0.9971✓

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Question 15 (8 marks)

(✓ = 1 mark)

a First equation: Find the area under curve using average height.

This is equivalent to the area of a rectangle of length $b - (-a) = a + b$ and width $\frac{3}{4}$.

Area 1 $= (a+b) \times \frac{3}{4} = \mathbf{\frac{3(a+b)}{4}}$ ✓

Second equation: Separately find the area.

First area is the area of the triangle with base a and height $2a$.

$= \frac{1}{2} \times a \times 2a = a^2$

Second area is the area of the trapezium with parallel sides b and $2a$ and width b.

$\mathbf{= \frac{1}{2}(2a+b)b}$ ✓

Total of first and second areas is $\mathbf{a^2 + \frac{1}{2}(2a+b)b}$ ✓

Each total area is 1.

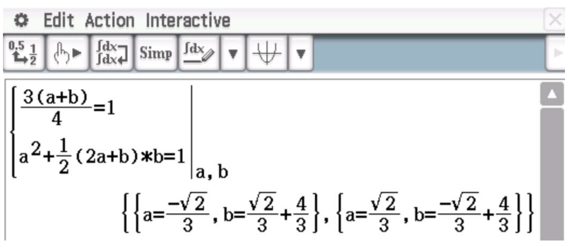
We now have two equations to solve for a and b.

$\mathbf{\frac{3(a+b)}{4} = 1 \text{ and } a^2 + \frac{1}{2}(2a+b)b = 1}$ ✓

b Use CAS to solve for a and b.

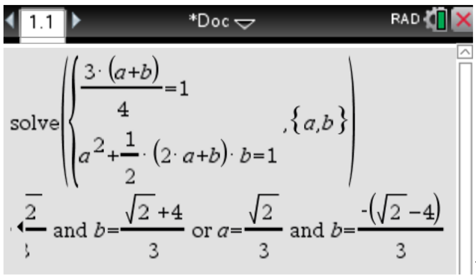
$\mathbf{\frac{3(a+b)}{4} = 1 \text{ and } a^2 + \frac{1}{2}(2a+b)b = 1}$ ✓

ClassPad



✓

TI-Nspire



✓

Since a and b are both positive values, $\mathbf{a = \frac{\sqrt{2}}{3}, b = \frac{4}{3} - \frac{\sqrt{2}}{3}}$ ✓

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c Use CAS to solve for a and b.

$\mathbf{P(X > 0) = 1 - P(X \le 0)}$ ✓

$\mathrm{P}(X \le 0)$ corresponds to the area of the triangle to the left of the y-axis.

$$= \frac{1}{2} \times a \times 2a = a^2$$

The area is $= \left(\frac{\sqrt{2}}{3}\right)^2$

$$= \frac{2}{9}$$

Hence $\boldsymbol{P(X > 0) = 1 - \frac{2}{9} = \frac{7}{9}}$ ✓

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EXERCISE 8.4 The normal distribution

Question 1

The distribution is symmetrical, so the mean is the midpoint.

$$\frac{50+150}{2}=100$$

The correct response is **B**.

Question 2

The maximum value corresponds to the height of the triangle.

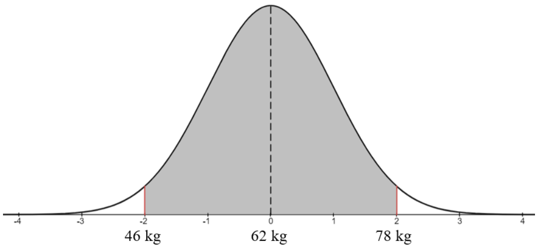
distribution is symmetrical, so the mean is the midpoint.

$$\frac{1}{2}\times 100\times height=1\Rightarrow height=0.02$$

The correct response is **B**.

Question 3

a

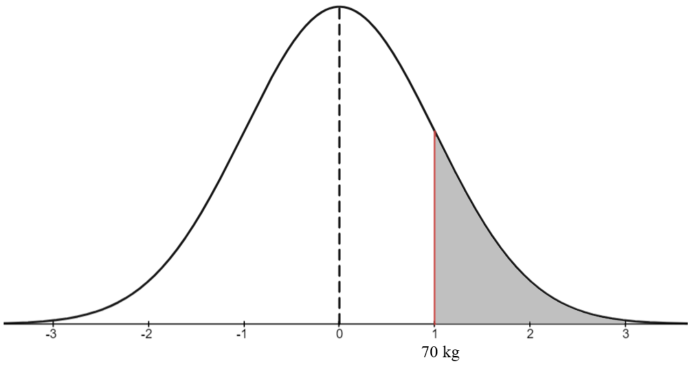


$P(46 \le X \le 78) = P(62 - 2\times 8 \le X \le 62 + 2\times 8)$

There are 2 standard deviations on either side of the mean. This represents 95% of the total probability.

$P(46 \le X \le 78) =$ **0.95**

b



$P(X > 70) = P(X > 62 + 8)$

8 means 1 standard deviation from the mean. This represents $\frac{0.68}{2} = 0.34$ of the total probability. So $P(X > 70) = 0.5 - 0.34 = \mathbf{0.16}$

c $P(X \le 38) = P(X \le 62 - 3\times 8)$

$62 - 3\times 8$ is 3 standard deviations below the mean. This represents $\frac{0.997}{2} = 0.4985$ of the total probability. So $P(X \le 38) = 0.5 - 0.4985 = \mathbf{0.0015}$

Question 4

a Using the 68-95-99.7% rule, three standard deviations will be $3 \times 700 = 2100$ kg.

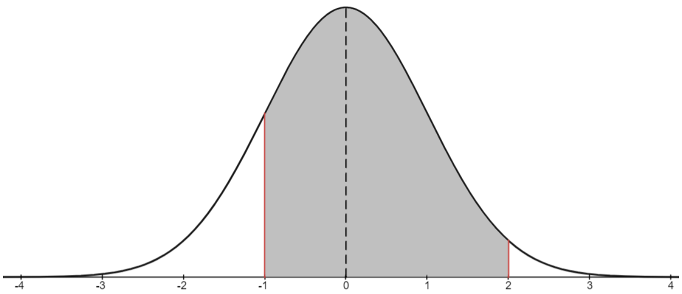
However, below the mean we have $1500 - 2100 = -600$, which gives a negative mass.

Hence this model is not appropriate. A mass of 0 kg is 2.14 standard deviations below the mean, so any higher values for the number of standard deviations will not make sense.

b The number of followers is a discrete variable, but the normal distribution is continuous. Hence the model is not appropriate.

Question 5

a

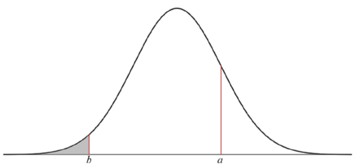


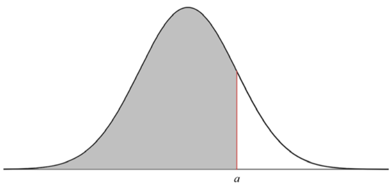
-1 is 1 standard deviation below the mean, $\dfrac{68}{2} = 34\%$.

2 is 2 standard deviations above the mean, $\dfrac{95}{2} = 47.5\%$.

Total is 34 + 47.5 = 81.5% = **0.815**

b





$\mathrm{P}(Z \le b) = 0.18$ $\mathrm{P}(Z \le a) = 0.82$ $\mathrm{P}(b \le Z \le a) = 0.82 - 0.18 = \mathbf{0.64}$

Question 6

a **i** $x = 14,\ \mu = 35,\ \sigma = 7\ \ z = \frac{x-\mu}{\sigma} = \frac{14-35}{7} = -3$

ii $x = 49,\ \mu = 35,\ \sigma = 7\ \ z = \frac{x-\mu}{\sigma} = \frac{49-35}{7} = 2$

iii $x = 59.5,\ \mu = 35,\ \sigma = 7\ \ z = \frac{x-\mu}{\sigma} = \frac{59.5-35}{7} = 3.5$

b $\mathrm{P}(X \geq 49) = \mathrm{P}(Z \geq 2) = 1 - \mathrm{P}(Z \leq -2)$

$= 1 - 0.975$

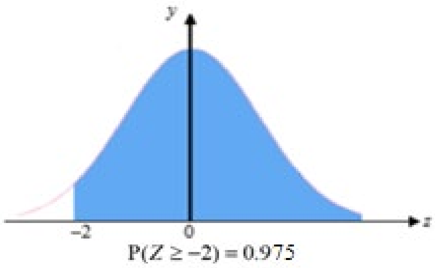
$= \mathbf{0.025}$

Alternatively.

We are told that $X \sim \mathrm{N}(35, 7^2)$

We are told that $\mathrm{P}(Z \geq -2) = 0.975$ and requires $\mathrm{P}(X \geq 49)$.

Draw a normal distribution curve that illustrates $\mathrm{P}(Z \geq -2) = 0.975$.



Find z, the standard normal value for $x = 49$.

$x = 49, \mu = 35$ and $\sigma = 7$

$z = \frac{x-\mu}{\sigma}$

$= \frac{49-35}{7}$

$= 2$

Hence, $\mathrm{P}(X \geq 49) = \mathrm{P}(Z \geq 2)$.

Find $\mathrm{P}(X \geq 49)$ using the equivalent area from a standard normal curve.

Draw the required area on a normal curve and calculate by symmetry.

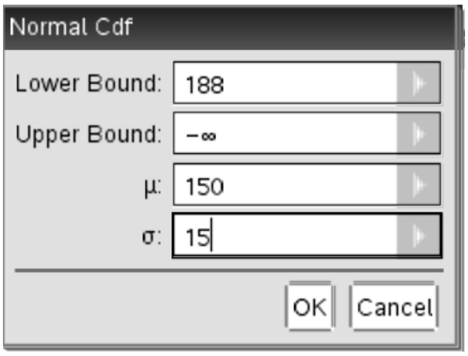
We know that $\mathrm{P}(Z \geq -2) = 0.975 = \mathrm{P}(Z \leq 2)$ by the symmetry of the normal distribution.

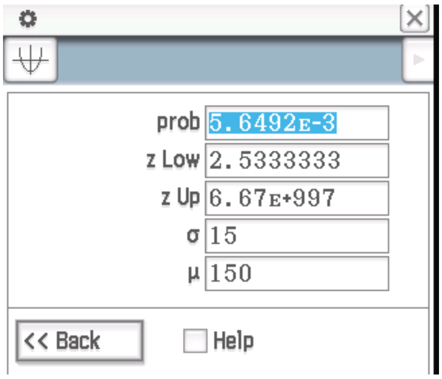
Also, $\mathrm{P}(Z \geq 2) = 1 - \mathrm{P}(Z \leq 2) = 1 - 0.975 = 0.025$.

Hence, $\mathrm{P}(X \geq 49) = \mathbf{0.025}$.

Question 7

a ClassPad





P(X > 188) = 0.0056

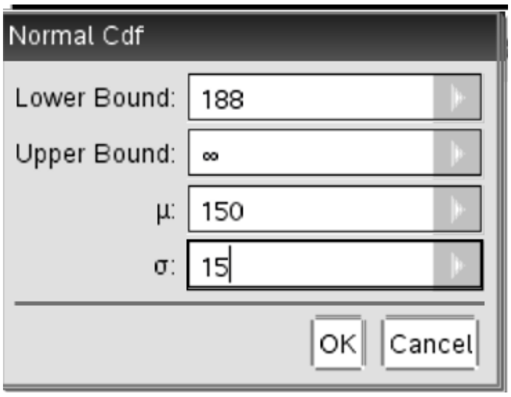
1 Tap **Interactive** > **Distribution/Inv. Dist** > **Continuous** > **normCDf**.

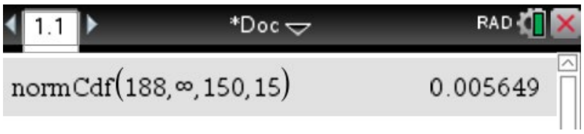
2 In the dialogue box, enter the corresponding lower, upper, σ and μ values.

3 Tap **OK** and the probability will be displayed.

(Always use -∞ or ∞ from the **Math2** keyboard for the 'ends' of the normal distribution.)

TI-Nspire



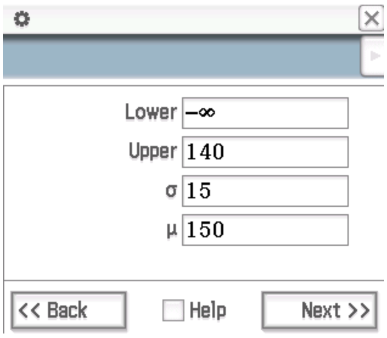


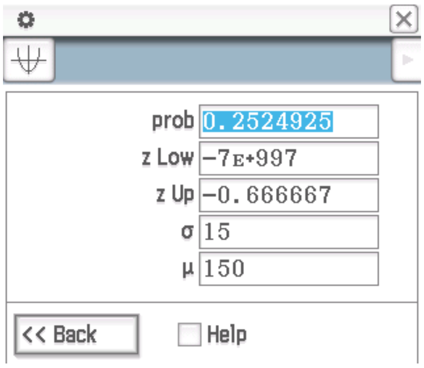
P(X > 188) = 0.0056

1 Press **menu** > **Probability** > **Distributions** > **Normal Cdf**.

2 In the dialogue box, enter the corresponding lower bound, upper bound, μ and σ.

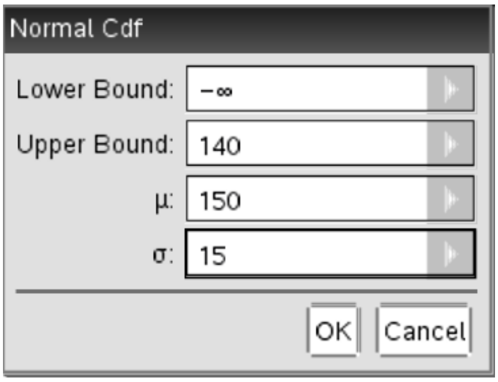
b ClassPad

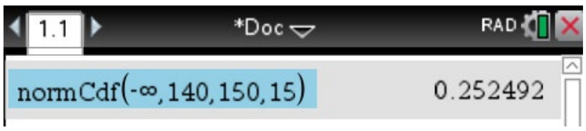




$\mathrm{P}(X \le 140) = 0.2525$

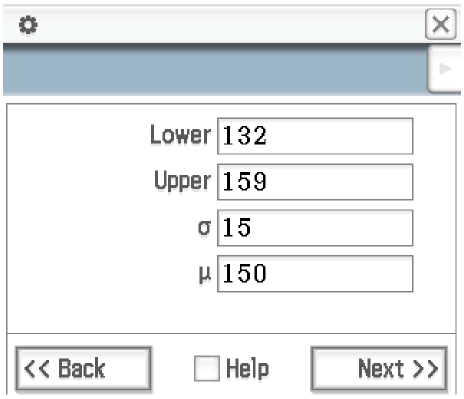
TI-Nspire

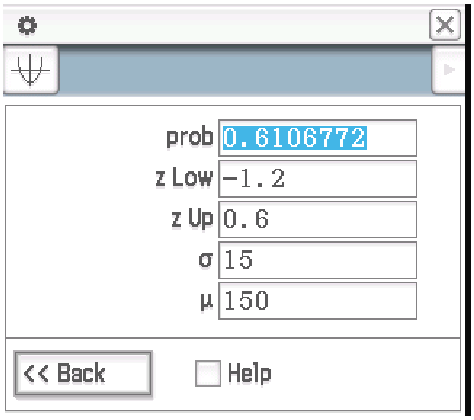




$\mathrm{P}(X \le 140) = 0.2525$

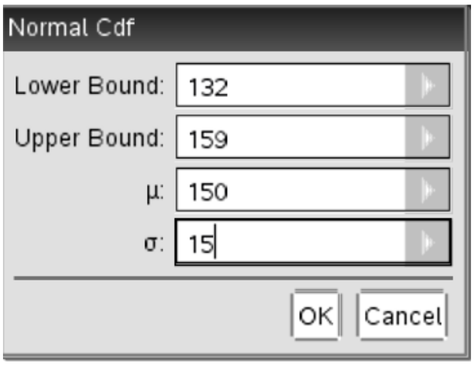
c ClassPad

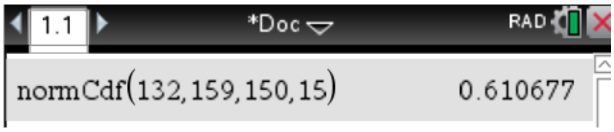




$\mathrm{P}(132 < X < 159) = 0.6107$

TI-Nspire

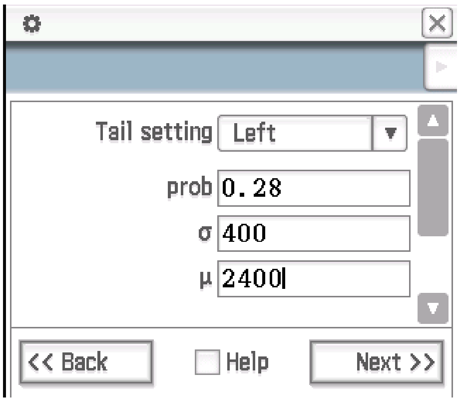


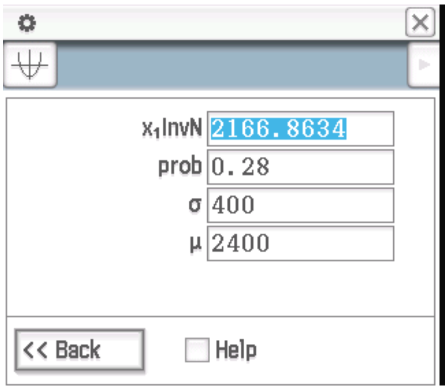


$\mathrm{P}(132 < X < 159) = 0.6107$

Question 8

a **ClassPad**

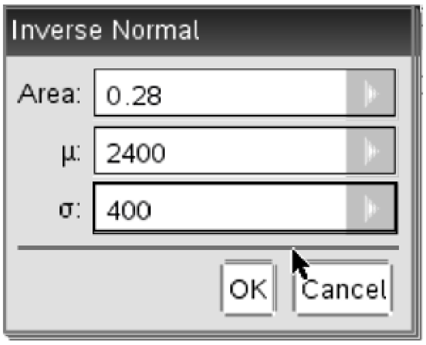




$c = 2166.86$

1 Tap **Interactive** > **Distribution/Inv.Dist** > **Inverse** > **invNormCDf**.
2 In the dialogue box, use the default **Tail setting** as **Left** for < or ≤.
3 Enter the values as shown above.
4 The answer will be displayed.

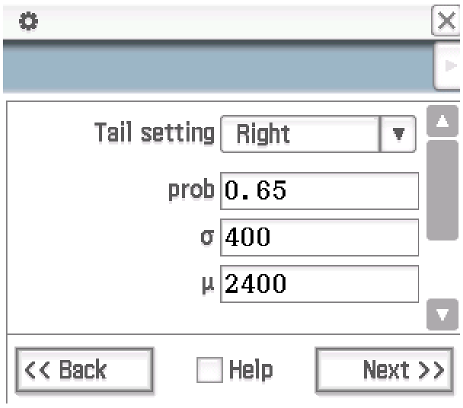
TI-Nspire

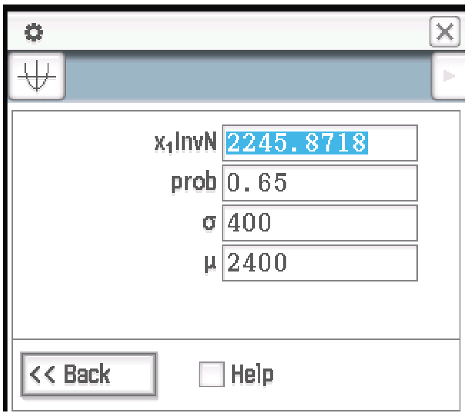


$c = 2166.86$

1 Press **menu** > **Probability** > **Distributions** > **Inverse Normal.**
2 In the dialogue box, enter the values as shown above.
3 The answer will be displayed.

b **ClassPad**



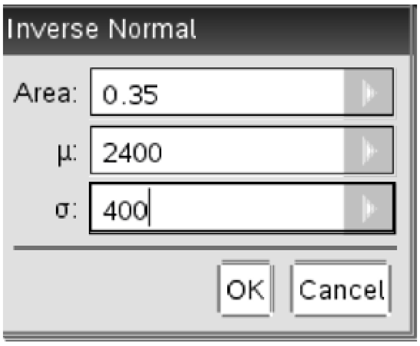


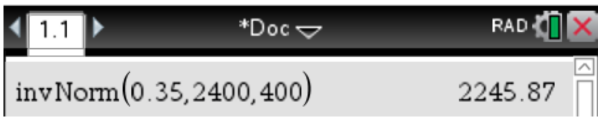
***c* = 2245.87**

1 Tap **Interactive** > **Distribution/Inv.Dist** > **Inverse** > **invNormCDf**.

2 In the dialogue box, use the default **Tail setting** as **Right**

TI-Nspire





***c* = 2245.87**

1 Press **menu** > **Probability** > **Distributions** >**Inverse Normal.**

2 In the dialogue box, enter the values as shown above.

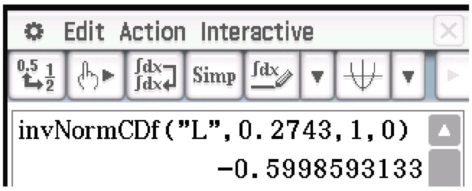
Note that the probability used is 1 – 0.65.

3 The answer will be displayed.

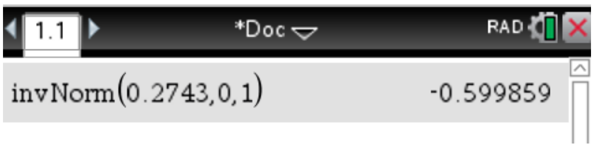
Question 9

Find the c value using $z = 1$, $\mu = 0$ and $P(Z < c) = 0.2743$.

ClassPad



TI-Nspire



$c = -0.59985\ldots$

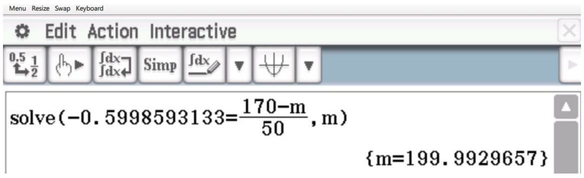
Now use $x = 170$ and $z = \dfrac{x-\mu}{\sigma}$ to solve for μ.

$$-0.5999 = \frac{170-\mu}{50}$$

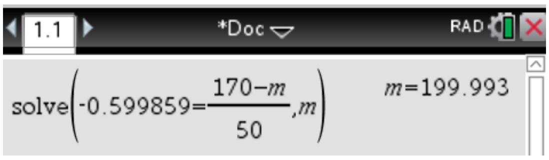
$$\mu = 50 \times 0.5999 + 170$$

$$\boldsymbol{\mu \approx 200}$$

ClassPad



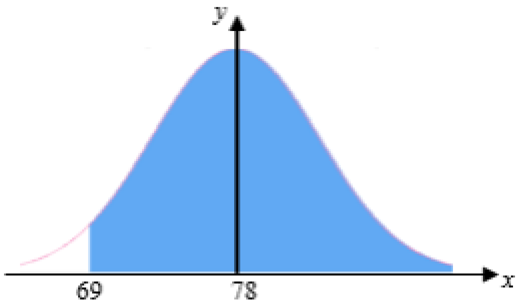
TI-Nspire



Question 10

a Let X be the random variable representing the weight of an egg from the Clucky Hen Egg Farm. We are told that $X \sim N\left(78, 6^2\right)$ and requires P(X > 69).

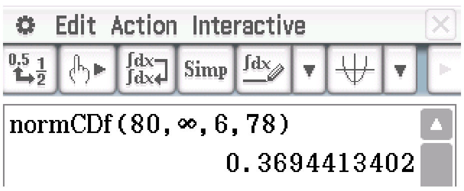
Draw a normal distribution curve that illustrates P(X ≥> 69).



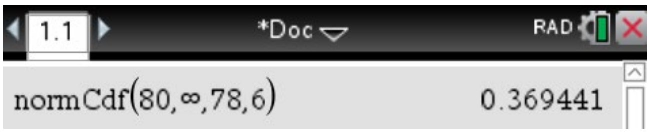
Using CAS, we obtain P(X > 69) ≈ **0.9332** to four decimal places.

b Find P(X >80)

ClassPad



TI-Nspire



P(X >80) = 0.3694

c Require $\text{P}\left(X > 80 \mid X > 69\right)$

$$\begin{aligned}\text{P}\left(X > 80 \mid X > 69\right) &= \frac{\text{P}\left(X > 80\right)}{\text{P}\left(X > 69\right)} \\ &= \frac{0.369441...}{0.933192...} \\ &= 0.395890... \\ &\approx \mathbf{0.3959}\end{aligned}$$

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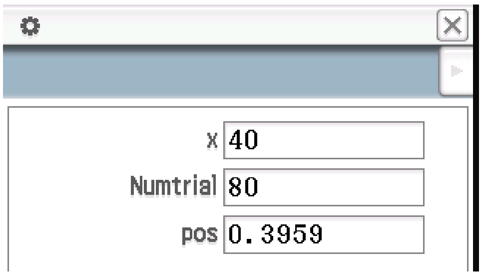
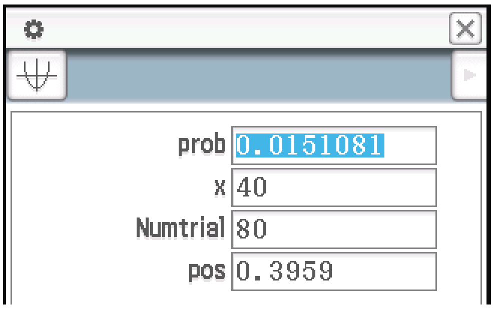
d Let X be the number of eggs weighing more than 69 g being classified as Jumbo.

This is a binomial distribution, $X \sim \text{Bin}(80, 0.3959)$.

$P(X = 40) = 0.0151$

ClassPad

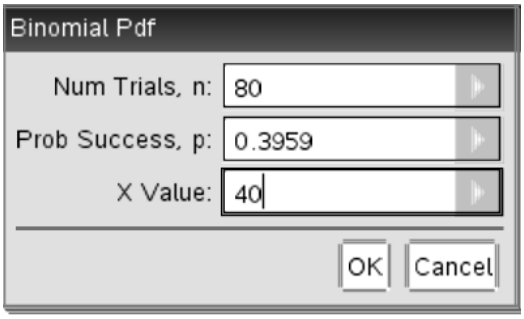


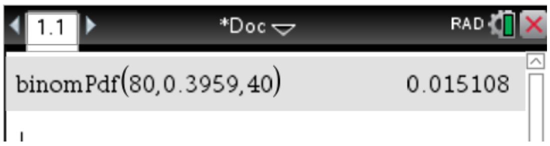


1 In the Statistics menu choose **Choose Calc** > **Distribution** > **Binomial PDf**

2 Enter the values shown and tap NEXT

TI-Nspire





1 From menu, choose **Statistics** > **Distributions** > **Binomial Pdf**

2 Enter the values shown and tap OK.

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Question 11 (3 marks)

(✓ = 1 mark)

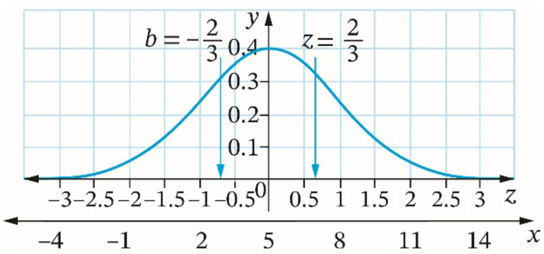
a Since 5 is the mean, and divides the distribution into two equal parts, $P(X > 5) = \mathbf{0.5}$ or $\mathbf{\frac{1}{2}}$. ✓

b We require b such that $P(X > 7) = P(Z < b)$.

As $X \sim N(5,\ 3^2)$ with $\sigma = 3$, calculate the value of Z associated with $x = 7$.

$z = \frac{x-\mu}{\sigma} = \frac{7-5}{3} = \mathbf{\frac{2}{3}}$ ✓

Hence, $P(X > 7) = P\left(z > \frac{2}{3}\right) = P\left(z < -\frac{2}{3}\right)$ by the symmetry of the distribution and $\boldsymbol{b = -\frac{2}{3}}$. ✓



Question 12 (5 marks)

(✓ = 1 mark)

a The mean is 2.5 and the standard deviation is 0.3.

$2.5 + 2 \times 0.3 = 3.1$ so $\text{P}(X > 3.1)$ will be the probability greater than 2 standard deviations from the mean, which is $\mathbf{P}(\boldsymbol{Z} > \mathbf{2})$.✓

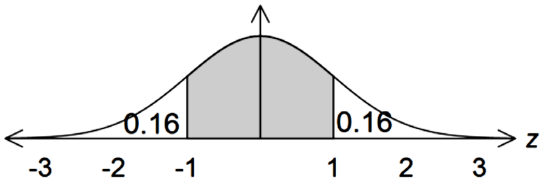
Using reflection of the normal distribution curve, this is equal to $\text{P}(Z < -2)$. Therefore $\boldsymbol{b} = \mathbf{-2}$.✓

b We are given that $\text{P}(Z < -1) = 0.16$ so by symmetry, $\text{P}(Z > 1) = 0.16$.

The mean $\mu = 2.5$ so $\text{P}(X > 2.5) = 0.5$.

The standard deviation $\sigma = 0.3$ so $\mu + \sigma = 2.5 + 0.3 = 2.8$.

Therefore, $\mathbf{P}(\boldsymbol{X} > \mathbf{2.8}) = \mathbf{P}(\boldsymbol{Z} > \mathbf{1}) = \mathbf{0.16}$.✓



Using conditional probability,

$$\begin{aligned}\text{P}(X < 2.8 \mid X > 2.5) &= \frac{\text{P}(X < 2.8 \cap X > 2.5)}{\text{P}(X > 2.5)} \\ &= \frac{\mathbf{P}(\mathbf{2.5} < \boldsymbol{X} < \mathbf{2.8})}{\mathbf{P}(\boldsymbol{X} > \mathbf{2.5})} ✓ \\ &= \frac{0.5 - 0.16}{0.5} \\ &= \mathbf{0.68} ✓\end{aligned}$$

Question 13 (6 marks)

(✓ = 1 mark)

a Let $X \sim N\left(72,\ 8^2\right)$.

The z value corresponding to $X = 80$ is $z = \dfrac{x-\mu}{\sigma} = \dfrac{80-72}{8} = \mathbf{1}$.✓

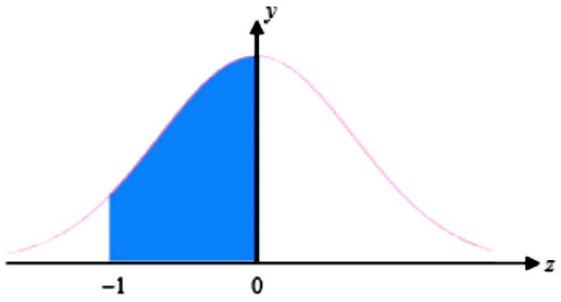
Hence P(X > 80) = P(Z > 1) = 1 – P(Z < 1) = 1 – 0.84 = **0.16**.✓

b The z value for x = 64 is $z = \dfrac{x-\mu}{\sigma} = \dfrac{64-72}{8} = -1$

Since the mean is 72, the z value of x = 72 is 0.

∴ **P(64 < *X* < 72) = P(–1 < *Z* < 0)**.✓

Draw a diagram of this event for the standard normal distribution.



P(Z < 0) = 0.5

From part **a**, P(Z < –1) = P (Z > 1) = 0.16.

∴ P (–1 < Z < 0) = P(Z < 0) – P(Z < –1) = 0.5 – 0.16 = **0.34**.✓

Examination report

Success percentage: 45%

The most common incorrect responses were 0.32 and 0.68.

c $$\mathrm{P}\left(X<64 \mid X<72\right)=\frac{\mathrm{P}\left(X<64 \cap X<72\right)}{\mathrm{P}\left(X<\ 72\right)}=\frac{\mathbf{P}\left(\boldsymbol{X}<\mathbf{64}\right)}{\mathbf{P}\left(\boldsymbol{X}<\mathbf{72}\right)}✓$$

$$=\frac{\mathrm{P}\left(Z<-1\right)}{\mathrm{P}\left(Z<\ 0\right)}=\frac{0.16}{0.5}=\mathbf{0.32}\left(\mathbf{or}\ \frac{\mathbf{8}}{\mathbf{25}}\right)✓$$

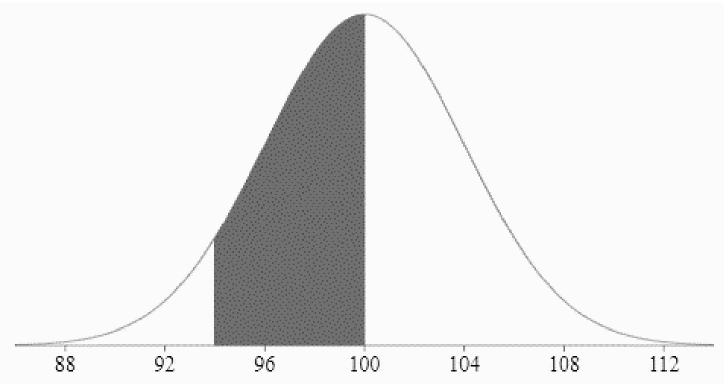
Question 14 (2 marks)

(✓ = 1 mark)

We know that $X \sim N\left(100,\ 4^2\right)$ and that P(X < 106) = q.

We require P(94 < X < 100) in terms of q.

Draw the required event as the area under the normal distribution for X.



Given P(X < 106) = q, then P(X > 106) = 1 – q.

By symmetry of the distribution, P(X < 94) = 1 – q also (note that X = 106 is 1.5 standard deviations above the mean and X = 94 is 1.5 standard deviations below the mean of the distribution).

By definition, P(X < 100) = 0.5.

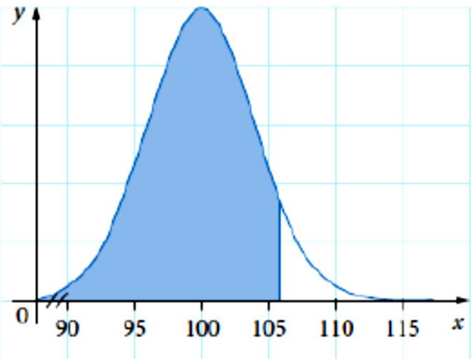
Hence, P(X < 100) = $\mathbf{\frac{1}{2}}$ **= P(X < 94) + P(94 < X < 100)✓**

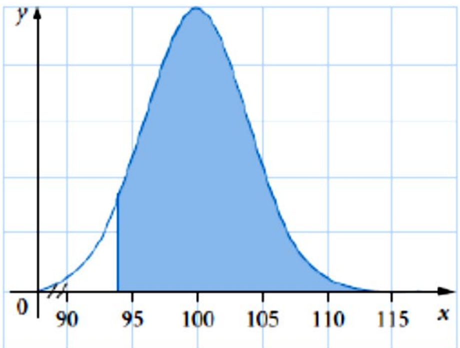
$$\Rightarrow \text{P}(94 < X < 100) = \frac{1}{2} - \text{P}(X < 94)$$

$$= \frac{1}{2} - (1 - q)$$

$$= q - \frac{1}{2}$$

$$= \boldsymbol{q - 0.5}\checkmark$$





Alternatively, P(X < 106) = q is the same area as P(X > 94), and as. P(X > 100) = 0.5, P(94 < X <100) = q – 0.5.

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Question 15 **[SCSA MM2018 Q2 MODIFIED] (6 marks)**

(✓ = 1 mark)

a Her statement is valid as 142 cm is three standard deviations below the mean and 184 cm is three standard deviations above the mean. Therefore, approximately 99.7% of women will have heights in that range, i.e. almost all.

states that the comment is appropriate✓

refers to the standard deviation and 99.7%✓

b 170 – 163 = 7 which is 1 standard deviation above.

$$\text{Percentage} = \frac{100-68}{2} = 16\%$$

states 1 standard deviation above✓

determines correct percentage✓

c percentage = 100 – 2 × 2.5

= 95%

2 SDs below = 163 – 14

= 149 cm

determines 95%✓

states height✓

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Question 16 [SCSA MM2021 Q6] (7 marks)

(✓ = 1 mark)

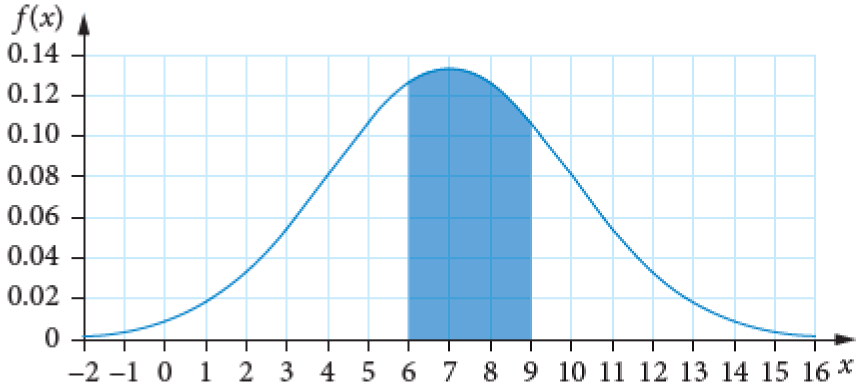
a **i** Mean = 2

determines mean of A✓

ii C has the largest standard deviation as it is the widest distribution.

states that C has the largest standard deviation and provides correct justification✓

b **i**



shades correct region✓

ii No. The total area below the probability density function is 1, and the region shaded above is less than half of that area (i.e. area is less than 0.5). Hence, it corresponds to a probability that is less than 0.5.

states that the probability is not greater than or equal to 0.5✓

provides correct justification✓

c Not normal: a continuous random variable has $P(Y \geq 2) = P(Y > 2)$. Since a normally distributed random variable is continuous it follows that Y is not a normally distributed random variable.

Could be binomial: $P(Y \geq 2) > P(Y > 2)$ for a discrete random variable. Since the binomial distribution is discrete it follows that Y could be a binomially distributed random variable.

states that *Y* could not be normal and provides a correct explanation✓

states that *Y* could be binomial and provides a correct explanation✓

Question 17 (5 marks)

(✓ = 1 mark)

a $X \sim N\left(200, \sigma^2\right)$

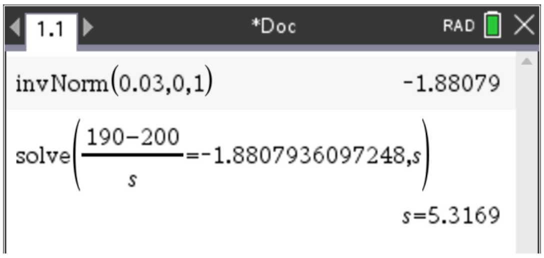
97% of the lollies weigh more than 190 g.

$\mathrm{P}(X > 190) = 0.97$ so $\mathbf{P}(X \leq 190) = \mathbf{0.03}$.✓

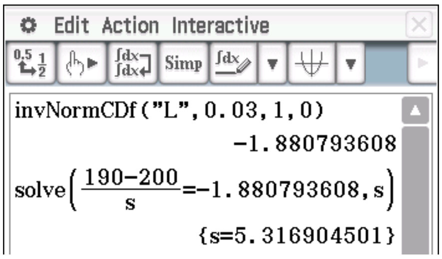
The inverse normal value of 0.03 for $\mu = 0$ and $\sigma = 1$ is -1.88079.

To determine the standard deviation, **solve** $\dfrac{190-200}{\sigma} = -1.88078$ **for** σ.✓

TI-Nspire



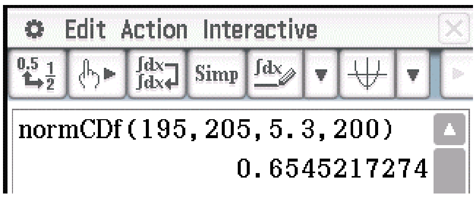
ClassPad



The standard deviation is 5.3 g.✓

b **TI-Nspire**

ClassPad



$\mathrm{P}(195 < X < 205) = 0.6545$

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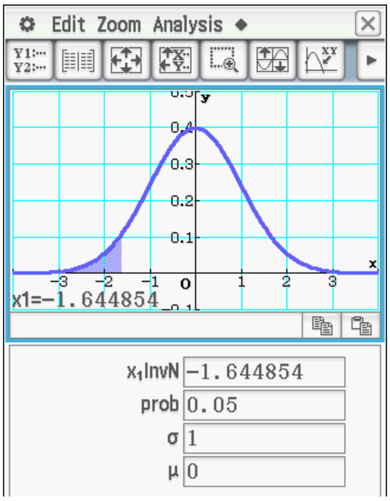
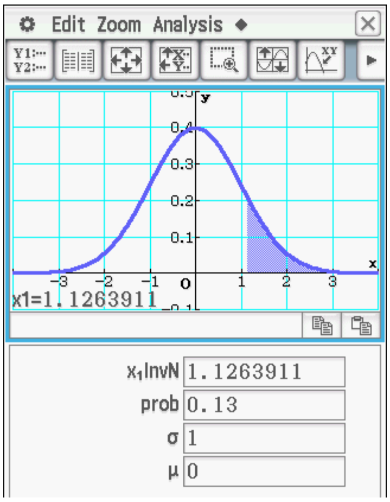
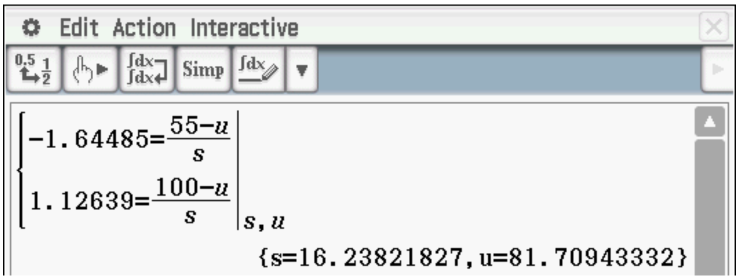
Question 18 **[SCSA MM2016 Q18] (6 marks)**

(✓ = 1 mark)

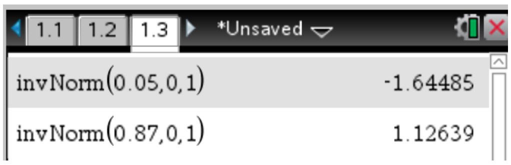
a **ClassPad**

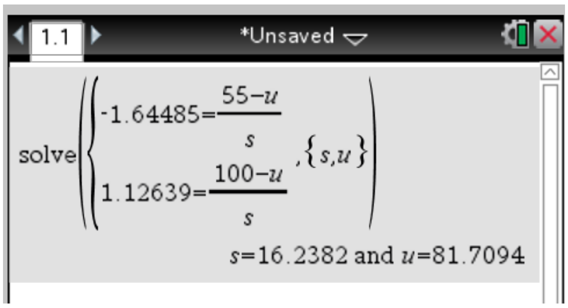






TI-Nspire





Find the 2 z-values.

Solve for mean and standard deviation.

So the mean is 81.71 minutes and the standard deviation is 16.24 minutes.

determines both z scores✓

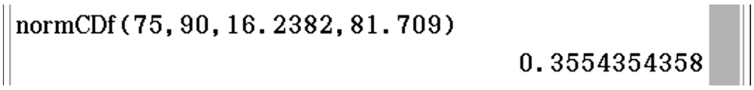
sets up at least one equation with mean and standard deviation✓

sets up two equations with mean and standard deviation✓

solves for mean✓

solves for standard deviation✓

b **ClassPad**



TI-Nspire

So, the probability that the waiting time will be between 75 and 90 minutes is 0.36.

determines probability✓

Question 19 [SCSA MM2020 Q8 MODIFIED] (7 marks)

(✓ = 1 mark)

a $P(X > 67) = 0.0808$ (or 8.08%)

states the correct expression for the probability✓

calculates the probability✓

b $$P(X < 75 \mid X > 67) = \frac{P(67 < X < 75)}{P(X > 67)} = \frac{0.0794}{0.0808} = 0.9832 \text{ (or } 98.32\%)$$

writes a conditional probability statement✓

recognises the restricted domain for 'jumbo' eggs✓

calculates the probability✓

c $P(X > m) = 0.0005$

$m = 76.45$ g

writes the correct expression $P(X > m) = 0.0005$✓

calculates the minimum weight✓

Question 20 **[SCSA MM2019 Q11] (8 marks)**

(✓ = 1 mark)

a $P(T>30)=P\left(Z>\dfrac{30-25}{2}\right)=P(Z>2.5)=1-0.9938=0.0062$

gives the correct value of the probability✓

b Let X denote the number of pizzas out of 50 that are delivered free. Then $X \sim \text{Bin}(50, 0.0062)$
$P(X > 3) = 1 - P(X \le 3) = 1 - 0.9997 = 0.0003$

states the distribution of the number of pizzas delivered free✓

computes the probability correctly✓

c $P(Z > z) = 0.001 \Rightarrow z = 3.0902 \Rightarrow t = 25 + 3.0902 \times 2 = 31.2$ minutes

uses a tail probability of 0.001✓

calculates the correct value of the delivery time✓

d $z=\dfrac{30-25}{\sigma}=3.0902 \Rightarrow \sigma=\dfrac{30-25}{3.0902}=1.62$ minutes

uses the correct critical value of the normal distribution✓

forms the correct equation for σ✓

solves for σ✓

Question 21 **[SCSA MM2021 Q8] (9 marks)**

(✓ = 1 mark)

a $\text{P}(W>155)=0.3451$

$0.3451 \times 100 = 34.51\%$

obtains correct probability✓

obtains correct percentage✓

b

Classification	Small	Medium	Large	Extra large
Weight W (grams)	$W \le 110$	$110 < W \le 155$	$155 < W \le 210$	$W > 210$
$\text{P}(W)$	**0.1418**	0.5131	0.3310	**0.3451 – 0.3310 = 0.0141**

determines one correct probability✓

determines second correct probability✓

c $\mathrm{P}(\text{Small} \mid \text{not Medium}) = \dfrac{\mathrm{P}(\text{Small})}{\mathrm{P}(\text{not Medium})} = \dfrac{0.1418}{0.4869} = 0.2912$

determines correct denominator✓

determines correct numerator and obtains final answer✓

d Let the random variable Y denote the number of small carrots in a bag.

Then $Y \sim \mathrm{Bin}(12, 0.1418)$

We need

$\mathrm{P}(Y \leq 2) = 0.7637$

defines appropriate random variable and states the correct binomial distribution✓

states the correct probability statement✓

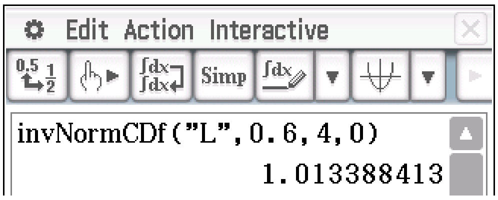
computes the probability✓

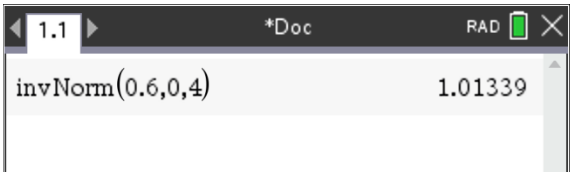
Question 22 (12 marks)

(✓ = 1 mark)

a Let $T \sim \mathrm{N}(0, 16)$

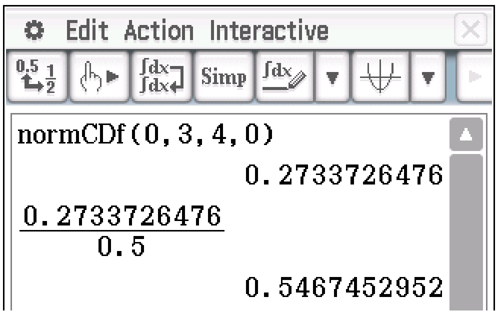
Use the inverse normal function to calculate $\mathrm{P}(T \leq a) = 0.6$.

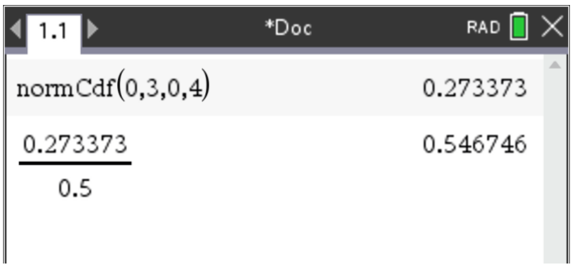




Rounding to the nearest minute, $\boldsymbol{a = 1}$ **minute.✓**

b Calculate $\mathbf{P}(T \leq 3 \mid T > 0) = \dfrac{\mathbf{P}(0 < T \leq 3)}{\mathbf{P}(T > 0)}$.✓



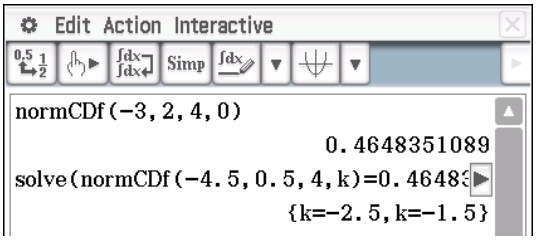


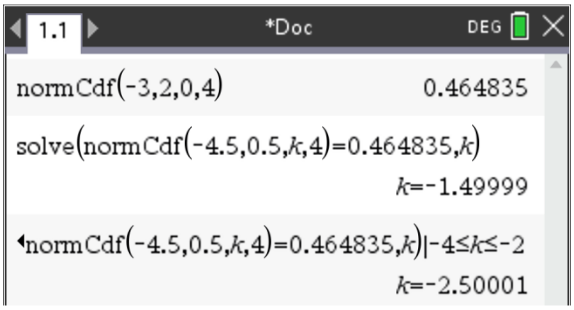
Rounding to four decimal places, the probability is **0.5467**.✓

c ***k* can be found by a translation of 1.5 units in the direction of the negative *t*-axis.✓**

Use the normCdf function and solve for *k*.

(TI users, restrict the domain to find second value of *k*.)



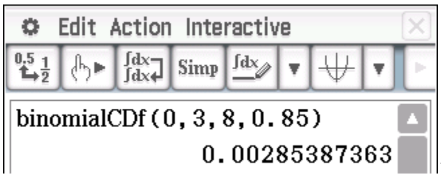


Alternatively, solve $\int_{-4.5}^{0.5}\left(\frac{1}{4\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{t-k}{4}\right)^2}\right)dt = 0.4648$ for *k*.

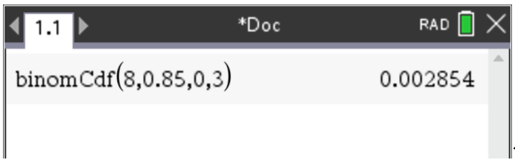
$\boldsymbol{k = -1.5}$ ✓ **or** $\boldsymbol{k = -2.5}$.✓

d This is a binomial distribution so let $X \sim \text{Bin}(8, 0.85)$.✓

Calculate $\text{P}(X \le 3)$



✓



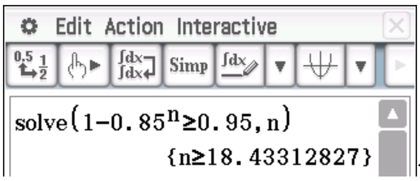
✓

Rounding to four decimal places, the probability is **0.0029**.✓

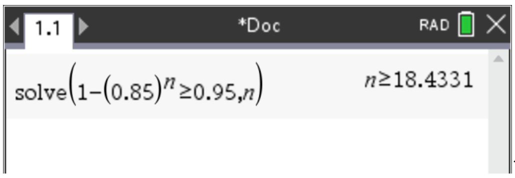
e **i** Let *X* be the probability that a delivery will not arrive on time.

$\text{P}(X \ge 1) = 1 - \text{P}(X = 0) = \mathbf{1 - 0.85^{\boldsymbol{n}}}$ ✓

ii Solve $1 - 0.85^n \ge 0.95$ for *n*.



✓



✓

Rounding up to the nearest whole number, $\boldsymbol{n = 19}$.✓

Cumulative examination: Calculator-free

Question 1 (4 marks)

(✓ = 1 mark)

a $\text{P}(\text{first ball is } 4) = \frac{1}{4}, \text{P}(\text{second ball is } 1) = \frac{1}{3}$

$\text{P}(\text{first ball is 4, second ball is } 1) = \frac{1}{4} \times \frac{1}{3} = \mathbf{\frac{1}{12}}$ ✓

b Possible outcomes that sum to 5 are 1 + 4, 4 + 1, 2 + 3 and 3 + 2.

The probability of each outcome is $\frac{1}{12}$, so the total probability of all the outcomes is

$\frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \mathbf{\frac{1}{3}}$ ✓

c Possible outcomes that sum to 5 are 1 + 4, 4 + 1, 2 + 3 and 3 + 2.

The favourable outcome is **4 + 1**.✓

The probability of this outcome is $\mathbf{\frac{1}{4}}$ ✓

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Question 2 (5 marks)

(✓ = 1 mark)

a

w	0	1	2	3
$P(W = w)$	0.2	0.4	0.3	0.1

✓

b **i** $E(W) = 0 \times 0.2 + 1 \times 0.4 + 2 \times 0.3 + 3 \times 0.1$ ✓

$= 1.3$ ✓

ii

w	$P(w)$	$w^2 \times P(w)$
0	0.2	0
1	0.4	0.4
2	0.3	1.2
3	0.1	0.9
Total	1.0	2.5

✓

$\text{Var}(W) = E(W^2) - E(W)^2$

$= 2.5 - 1.3^2$ ✓

$= 0.81$

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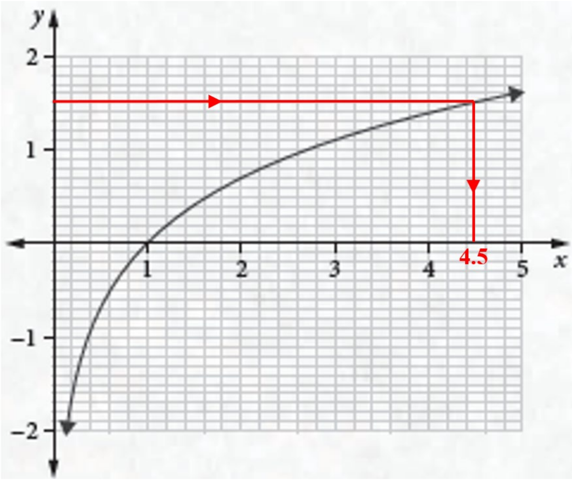
Question 3 (3 marks)

(✓ = 1 mark)

a



$x \approx 4.5$.✓

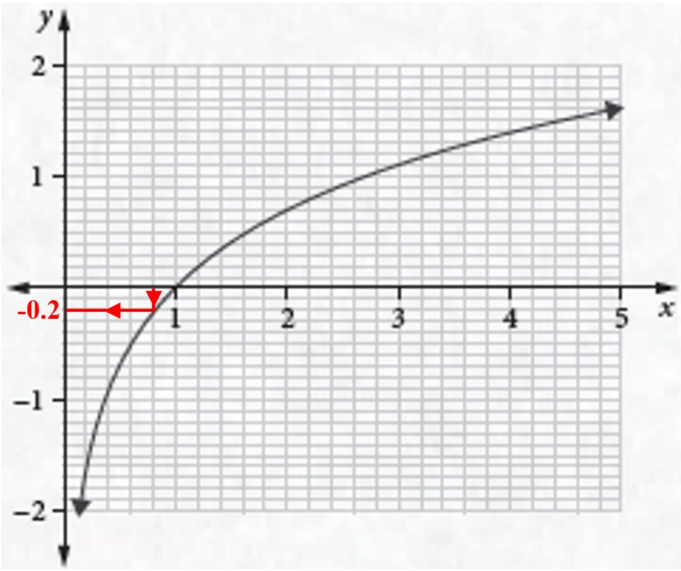
b

$e^{3-2x} = 0.8$

$3 - 2x = \ln(0.8)$

$x = \frac{1}{2}(3 - \ln(0.8)) = \frac{1}{2}(3 + 0.2)$

$x \approx 1.6$ ✓



Question 4 (7 marks)

(✓ = 1 mark)

a $\int 2\ln(x^4)dx = \int 8\ln(x)dx$

b $\int \frac{6x+15}{x^2+5x-11}dx = 3\int \frac{2x+5}{x^2+5x-11}dx$ ✓

$= 3\int \frac{f'(x)}{f(x)}dx$, where $f(x) = x^2+5x-11$.

Hence $\int \frac{6x+15}{x^2+5x-11}dx = \mathbf{3\ln\left(x^2+5x-11\right)+c}$ ✓

c $\int_1^3 \frac{3}{3x+1}dx$

$= \left[\ln(3x+1)\right]_1^3$ ✓

$= \ln(10) - \ln(4)$

$= \ln\left(\frac{10}{4}\right)$ ✓

$= \ln\left(\frac{5}{2}\right)$

$= \mathbf{\ln(5) - \ln(2)}$ ✓

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Question 5 (5 marks)

(✓ = 1 mark)

a $\frac{d}{dx}\left(x\sin\left(\frac{\pi}{4}x\right)\right) = x\frac{d}{dx}\left(\sin\left(\frac{\pi}{4}x\right)\right) + \frac{d}{dx}(x)\sin\left(\frac{\pi}{4}x\right)$, using product rule.✓

$$= \frac{\pi x}{4}\cos\left(\frac{\pi}{4}x\right) + \sin\left(\frac{\pi}{4}x\right) \checkmark$$

b $\left[x\sin\left(\frac{\pi x}{4}\right)\right]_0^2 = \int_0^2 \frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right)dx + \int_0^2 \sin\left(\frac{\pi x}{4}\right)dx$ ✓

$\left[x\sin\left(\frac{\pi x}{4}\right)\right]_0^2 = \int_0^2 \frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right)dx + \frac{4}{\pi}\left[-\cos\left(\frac{\pi x}{4}\right)\right]_0^2$ ✓

$2 = \int_0^2 \frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right)dx + \frac{4}{\pi}$

$\int_0^2 \frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right)dx = 2 - \frac{4}{\pi}$

Hence, $\mathbf{E}(X) = 2 - \frac{4}{\pi}$ ✓

Question 6 **[SCSA MM2021 Q2] (10 marks)**

(✓ = 1 mark)

a Continuous uniform distribution.

correctly states the name of the distribution✓

b i

$p = 15$

$q = 40$

$k = \frac{1}{25}$

correctly states the values of *p* and *q*✓

correctly states the values of *k*✓

ii

$$\mathrm{E}(X) = \frac{40+15}{2}$$
$$= 27.5 \text{ minutes}$$

correctly states the expected value✓

iii

$$\mathrm{P}(X < 25) = \frac{25-15}{25}$$
$$= \frac{10}{25}$$
$$= \frac{2}{5}$$

identifies the area between 15 and 25 is required✓

calculates the correct probability (simplified probability not required)✓

c

$$\mathrm{P}(X > 25 \mid X < 28) = \frac{3}{13}$$

correctly identifies the situation is a conditional probability✓

determines the correct probability✓

d

$$4\% = \frac{4}{100} = \frac{1}{25}$$

$\therefore$ leaves 39 minutes before 8:28 am

She should leave home at 7:49 am

determines 4% = a probability of $\frac{1}{25}$ ✓

correctly determines the time✓

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Cumulative examination: Calculator-assumed

Question 1 (9 marks)

(✓ = 1 mark)

a The surface area, SA, is given by

$$\boldsymbol{SA = 2\left(x \times \frac{5x}{2} + h \times x + h \times \frac{5x}{2}\right)} \checkmark$$

$$= 5x^2 + 2hx + 5hx$$

$$= 5x^2 + 7hx$$

Given $SA = 6480$,

$$5x^2 + 7hx = 6480$$

$$7hx = 6480 - 5x^2$$

$$\boldsymbol{h = \frac{6480 - 5x^2}{7x}} \checkmark$$

b Volume is positive, so we need $\boldsymbol{\frac{5x\left(6480 - 5x^2\right)}{14} > 0} \checkmark$

$6480 - 5x^2 > 0$, since $x > 0$

$$1296 - x^2 > 0$$

$$x^2 < 1296$$

$$\boldsymbol{0 < x < 36} \checkmark$$

c $V = \frac{5}{14}\left(6480x - 5x^3\right)$ ✓

$\frac{dV}{dx} = \frac{5}{14}\left(6480 - 15x^2\right)$ ✓

$= -\frac{75}{14}x^2 + \frac{16\,200}{7}$ ✓

d For maximum volume, $\frac{dV}{dx} = 0$.

$\frac{5}{14}\left(6480 - 15x^2\right) = 0$

$x^2 = \frac{6480}{15} = 432$

$x = \sqrt{432} = \sqrt{144 \times 3}$

$x = 12\sqrt{3}$ ✓

$h = \frac{6480 - 5x^2}{7x}$

$= \frac{6480 - 5\left(12\sqrt{3}\right)^2}{7\left(12\sqrt{3}\right)} = \frac{4320}{84\sqrt{3}}$

$h = \frac{120\sqrt{3}}{7}$ ✓

Question 2 [SCSA MM2020 Q15] (9 marks)

(✓ = 1 mark)

a $T(0) = 200 - 175e^{-0.07(0)}$

$= 25^0\text{C}$

states correct temperature✓

b $T(5) = 200 - 175e^{-0.07(5)}$

$= 76.68^0\text{C}$

states correct temperature✓

c $100 = T_0 - 175e^{-0.07(5)}$

$T_0 = 100 + 175e^{-0.07(5)}$

$\approx 223^0\text{C}$

states correct equation to be solved✓

solves for T_0, giving changed temperature✓

d $T'(t) = 12.25e^{-0.07t}$

$T'(5) = 12.25e^{-0.07(5)}$

$= 8.63°\text{C} / \text{min}$

states correct derivative of T with respect to t✓

calculates correct rate✓

e As time increases, the rate of change in the temperature of the water tends to 0.

The temperature of the water tends to the constant value of T_0.

states that the rate of change in the temperature tends to 0✓

states the water temperature approaches a constant✓

states the water temperature approaches 0✓

Question 3 [SCSA MM2020 Q10] (7 marks)

(✓ = 1 mark)

a

$$h'(2) = \frac{4(2)+1}{2(2)^2 + (2) + 1}$$

$$= \frac{9}{11} \text{ cm/s} \quad \{0.82\}$$

determines correct rate including units✓

b $h'(t)$ is of the form $\dfrac{f'(x)}{f(x)}$ (the numerator is the derivative of the denominator), so the function $h(t)$ is the natural logarithm of the denominator.

Also, $+c$ needs to be included in the function, as any constant can be included here.

states that the numerator is the derivative of the denominator✓

identifies the number c as the constant of integration✓

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c
$$\Delta h = \int_0^2 \frac{4t+1}{2t^2+t+1}\,dt$$
$$= \ln(11) \text{ cm} \quad \{2.40\}$$

determines total change✓

d Let the time taken for the bowl to be filled equals *a* seconds

$$\ln(56) = \int_0^a \frac{4t+1}{2t^2+t+1}\,dt$$
$$= \left[\ln\left(2t^2+t+1\right)\right]_0^a$$
$$= \ln\left(2a^2+a+1\right)$$
$$56 = 2a^2+a+1$$
$$a = 5$$

The bowl will take 5 seconds to completely fill.

states a definite integral for depth of water✓

equates definite integral to maximum water level✓

determines time taken✓

Question 4 [SCSA MM2017 Q19] (12 marks)

(✓ = 1 mark)

a

$$T \sim N\left(90, 15^2\right) \text{ so } P(T > 120) = P\left(Z > \frac{120-90}{15}\right) = P(Z > 2) = 0.0228$$

writes correct probability statement✓

calculates correct probability✓

b Let the random variable X denote the number of days out of 5 that the process takes more than 2 hours. Then $X \sim \text{Bin}(5, 0.0228)$.

$$P(X = 2) = \binom{5}{2}(0.0228)^2(1-0.0228)^3 = 0.00485$$

identifies binomial distribution✓

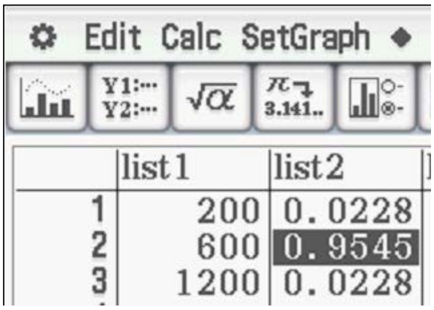
uses correct parameters for binomial✓

calculates correct probability✓

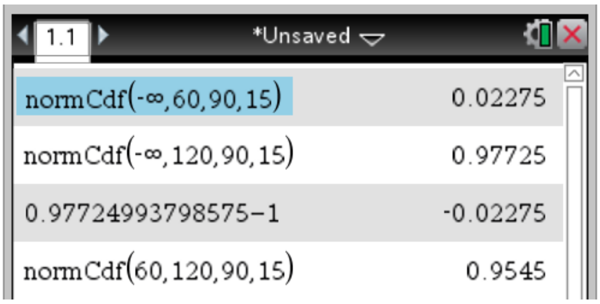
c **i**

Job duration T (minutes)	$T \le 60$	$60 < T < 120$	$T \ge 120$
Probability	0.0228	0.9545	0.0228
Cost Y ($)	200	600	1200

ClassPad

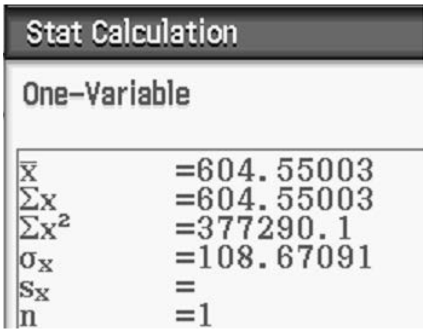


TI-Nspire

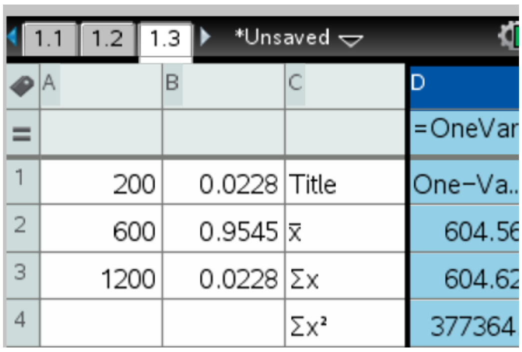


calculates the probabilities correctly✓

ii **ClassPad**



TI-Nspire



The probability distribution of Y is given below.

Mean cost = 200 × 0.0228 + 600 × 0.9545 + 1200 × 0.0228

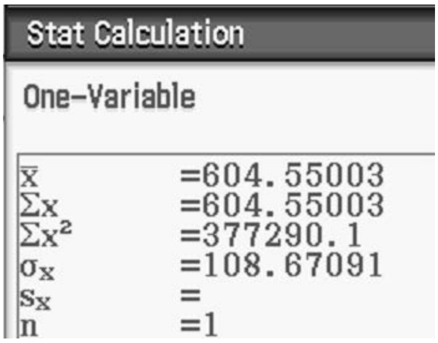
= $604.55

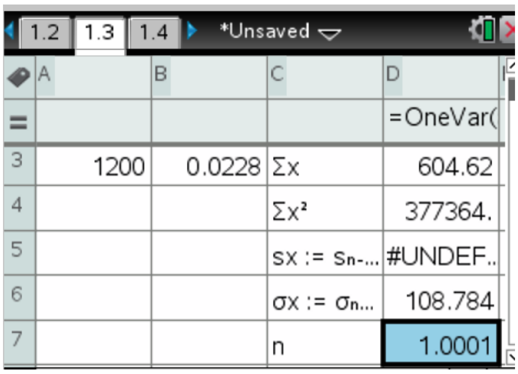
writes an expression for the mean cost per file✓

calculates the mean correctly✓

iii The probability distribution of Y is given below.

ClassPad **TI-Nspire**





$E\left(Y^2\right) = 200^2 \times 0.0228 + 600^2 \times 0.9545 + 1200^2 \times 0.0228 = 377\,290$

So $\text{Var}(Y) = 377\,290 - 604.55^2 = 11\,809.2975$ and $\sigma_x = \$108.67$

calculates $E(X^2)$ correctly✓

calculates the standard deviation correctly✓

iv New Mean = $604.55a + b$

New SD = $108.67a$

states new mean correctly✓

states new SD correctly✓

Question 5 [SCSA MM2018 Q12] (19 marks)

(✓ = 1 mark)

a $X \sim N(3,1)$

$P(X > 3.7) = 0.24196$

24.2% are greater than 3.7 kg

states weight required greater than 3.7 kg✓

obtains the correct percentage✓

b Let the random variable M denote the number of parcels that weigh more than 3.7 kg.

Then $M \sim \text{Bin}(20, 0.24196)$

$P(M > 10) = 0.01095$

states the distribution as binomial✓

determines the correct parameters of the distribution✓

obtains the correct probability✓

c

x	≤ 1	$1 < x \le 2$	$2 < x \le 3$	$3 < x \le 4$	$x > 4$
y	\$5	\$6.50	\$8	\$9.50	\$12
$P(Y = y)$	0.022 75 (accept 0.021 40)	0.135 91	0.341 34	0.341 34	0.158 66

obtains two correct values of *y*✓

obtains the other two correct values of *y*✓

obtains two correct probabilities✓

obtains the remaining correct probabilities✓

d $E(Y) = 5 \times 0.02275 + 6.5 \times 0.13591 + 8 \times 0.34134 + 9.50 \times 0.34134 + 12 \times 0.15866$

$= 8.874535$

That is, \$8.87 is the mean cost of postage per parcel.

obtains the correct expression for the mean✓

obtains the correct value of the mean✓

e $\sigma^2 = (5 - 8.87)^2 \times 0.02275 + (6.5 - 8.87)^2 \times 0.13591 + (8 - 8.87)^2 \times 0.34134 + (9.5 - 8.87)^2 \times 0.34134 + (12 - 8.87)^2 \times 0.15866$

$= 3.0523316$

$\sigma = 1.75$

substitutes into variance formula correctly✓

calculates the variance correctly✓

calculates the standard deviation correctly✓

f The mean will increase by 20% to $1.2 \times 8.87 + 1 = 11.64$.

The standard deviation increases by 20% to $1.2 \times 1.75 = 2.10$.

states new value will need to be multiplied by 1.2✓

correctly determines mean✓

correctly determines standard deviation✓

g There is a non-zero (small) probability that the weight can be negative, which is not possible.

calculates the probability of a weight below 0✓

explains that negative weights are not possible here✓

Chapter 9 – Interval estimates for proportions

EXERCISE 9.1 Random sampling

Question 1

a The population is **all pies produced at the factory**.

b There are six weight measurements, so **the sample size is 6**.

c Some possible statistics are:

mode = 110 g (the most frequent value)

range = 110 – 98 = 12 g

mean = (110 + 105 + 110 + 98 + 101 + 102) ÷ 6 ≈104.3 g

median = (102 +105) ÷ 2 = 103.5 g (middle value after numbers are listed in ascending order)

Question 2

Some possible methods:

1. Survey every 10^{th} customer as they check-out.
2. Randomly select rooms from all the occupied rooms.
3. Make the survey anonymous.

Other answers are possible.

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Question 3

a **i** Temporal bias. There might be bias in terms of the time of day or time of the year. For instance, business people who regularly fly might be more likely to catch early flights so that they can return on the same day. Holiday-makers are also more likely to catch flights outside business hours.

Spatial bias: people are being asked about flying at the airport and only one airport in the country is surveyed.

Other answers are possible.

ii Randomly select passengers at different times of the day and on different days. This will ensure that they are not from the same cohort, such as a group of footballers who regularly travel interstate for games.

Other answers are possible.

b **i** Leading question bias. The question “How many bags do you have?” might be interpreted in different ways. For instance, is it referring to shopping bags, personal bags, **or** reusable bags. Temporal bias: customers are only asked on one day for a one-hour window of time.

Other answers are possible.

ii Make the question unambiguous by asking, “How many bags did you bring with you to do your shopping at this store?” Ask every 15th customer in line at the self-service check-out across a larger time frame, or across multiple days, or across multiple stores.

Other answers are possible.

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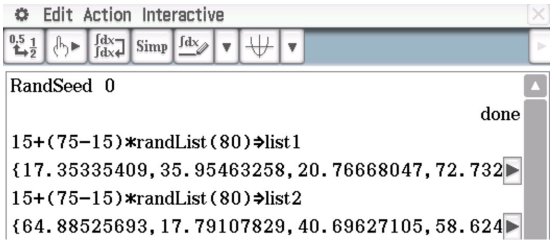
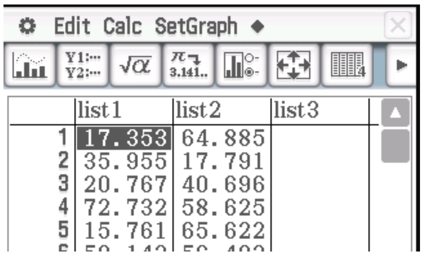
Question 4

Answers will vary depending on simulation.

ClassPad





1 Type **RandSeed** 0 (or choose **RandSeed** from the **Catalog** and add 0)

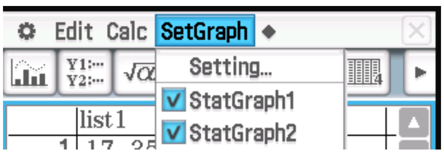
2 In **Catalog**, choose **randList** (or type **randList)**.

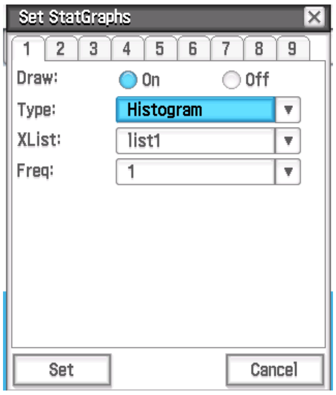
3 Enter and store the following formula in **list1**.

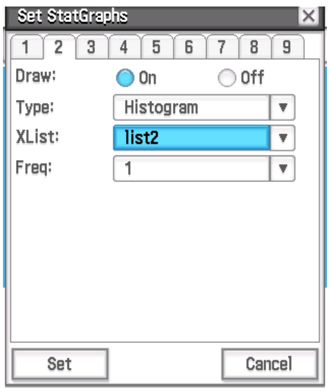
15 + (75 – 15)***randList**(80)⇨**list1**

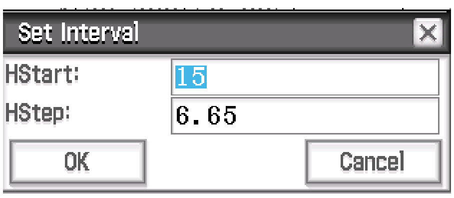
4 Repeat to obtain a second sample and store it in **list2**.

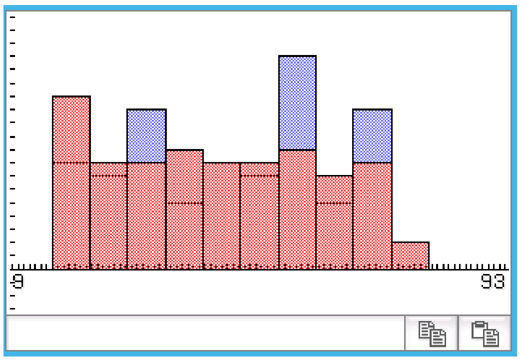
5 Tap **Menu** > **Statistics** to view the two lists.









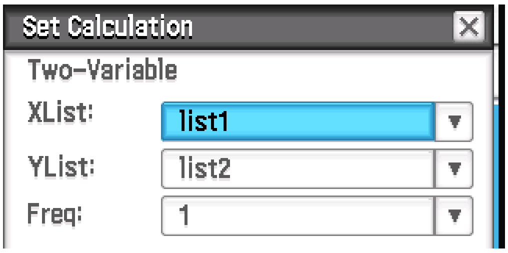


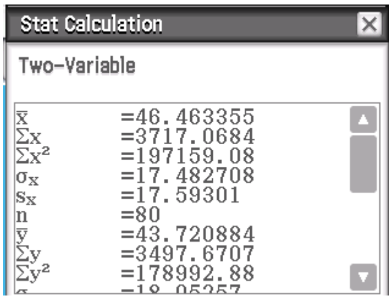
6 Tap **Setgraph** > **Setting**

7 Tap on the **1 tab** and change the **Type:** field to **Histogram** and the **XList** to **list1.**

8 Choose the 2 tab and change the **XList** to **list2.**

9 Tap Graph and keep the **HStart: field** as shown.





10 Tap **Calc > Two-Variable** and choose **list1 and list2**

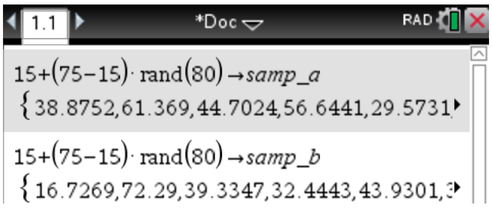
$$\mathrm{E}(X)=\frac{15+75}{2}=45,\ \mathrm{SD}(X)=\frac{75-15}{\sqrt{12}}=17.3$$

E(sample 1) ≈ 46.5, SD(sample 1) ≈ 17.5

E(sample 2) ≈ 43.7, SD(sample 2) ≈ 18.1

The second simulated mean is close to the expected mean, and the first simulated standard deviation is close to the expected standard deviation.

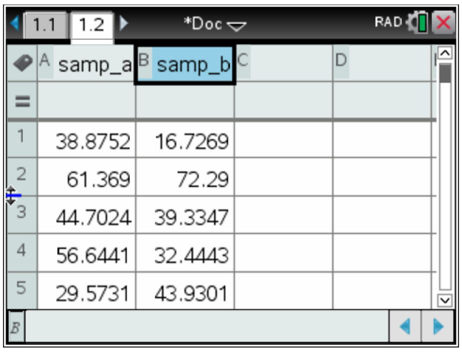
TI-Nspire



1 Type the information shown below to produce 80 random numbers stored as *samp_a*

15 + (75 – 15)***rand**(80)→*samp_a*

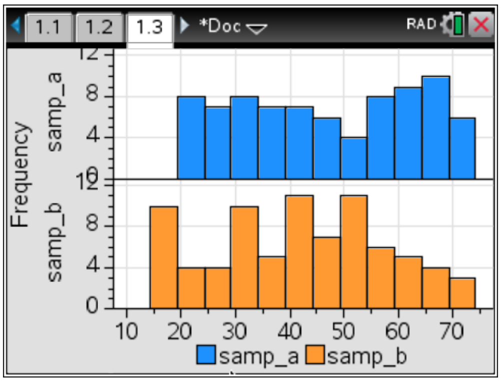
2 Repeat to produce a second set of random numbers stored as ***samp_b***.



3 Add a **Lists & Spreadsheet** page.

4 Tap in the cell next to the **A**, press **var** and select *samp_a*.

5 Tap in the cell next to the **B**, press **var** and select *samp_b*.



6 Add a **Data & Statistics** page.

7 For the horizontal axis, select *samp_a*.

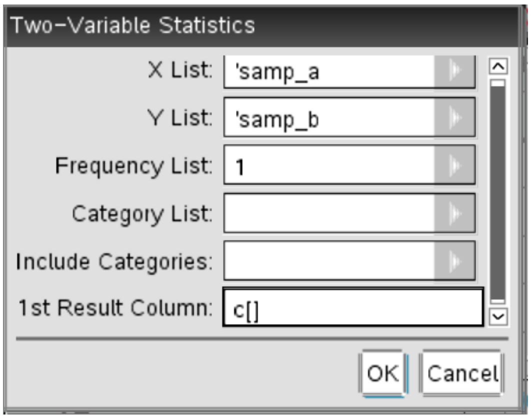
8 Press **menu > Plot Type > Histogram**.

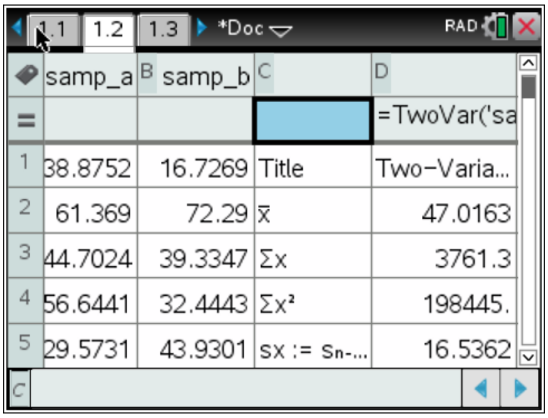
9 The sample a data will be displayed as a histogram.

10 Tap **menu** > **Plot Properties > Add X Variable**.

11 Select *samp_b*.

12 The data for both samples will be displayed as histograms.





13 Return to the **Lists & Spreadsheet page**.

14 Press **menu > Statistics > Stat Calculations >Two-Variable**.

15 In the **X List:** field, press the right arrow and select *samp_a*.

16 In the **Y List:** field, press the right arrow and select *samp_b*.

17 Press enter. The results are in columns **C** and **D**.

E(sample 1) ≈ 47.0, SD(sample 1) ≈ 16.5

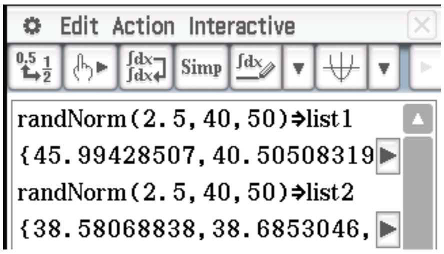
E(sample 2) ≈ 42.0, SD(sample 2) ≈ 15.8

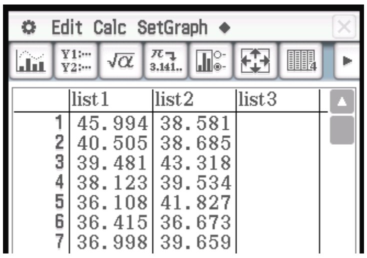
The first simulated mean is close to the expected mean, and the first simulated standard deviation is close to the expected standard deviation.

Question 5

Answers will vary depending on simulation.

ClassPad





1 Type **RandSeed** 0 (or choose **RandSeed** from the **Catalog** and add 0)

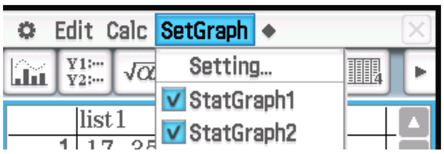
2 In **Catalog**, choose **randNorm (**or type **randNorm)**.

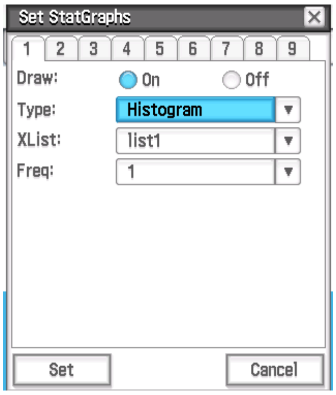
3 Enter and store the following formula in **list1**.

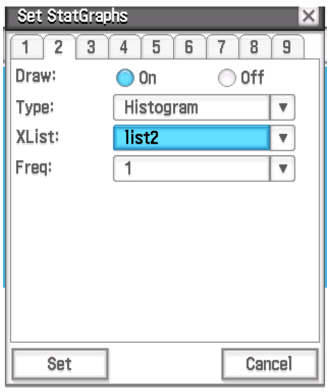
randNorm(2.5,40,50)⇨**list1**

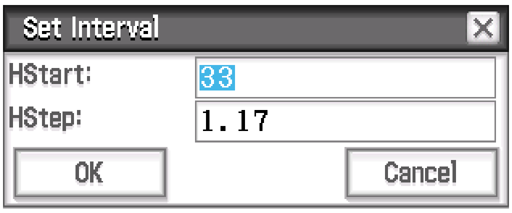
4 Repeat to obtain a second sample and store it in **list2**.

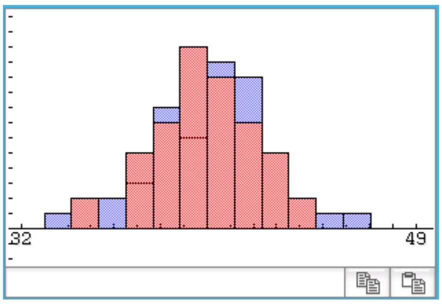
5 Tap **Menu** > **Statistics** to view the two lists.











6 Tap **Setgraph** > **Setting**

7 Tap on the **1 tab** and change the **Type:** field to **Histogram** and the **XList** to **list1.**

8 Choose the 2 tab and change the **XList** to **list2.**

9 Tap Graph and keep the **HStart: field** as shown.

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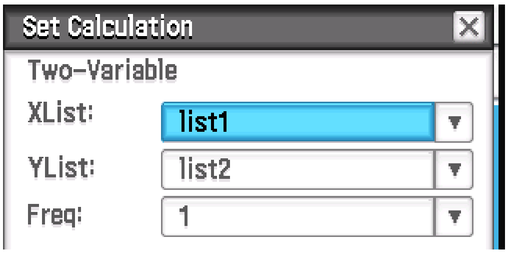
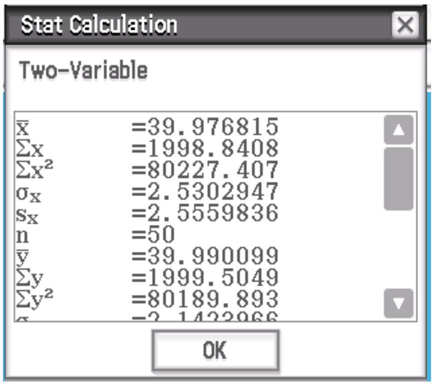




10 Tap **Calc>Two-Variable** and choose **list1 and list2**

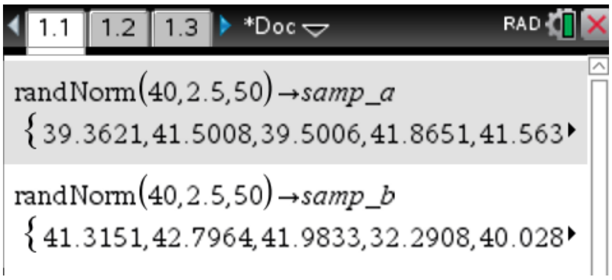
$\text{E}(X) = 40$, $\text{SD}(X) = 2.5$

E(sample 1) ≈ 40.0, SD(sample 1) ≈ 2.5

E(sample 2) ≈ 40.0, SD(sample 2) ≈ 2.1

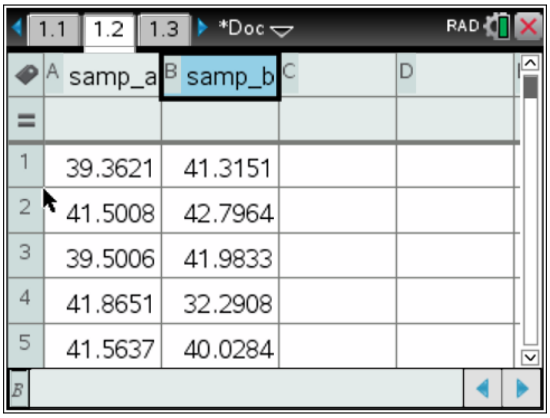
The simulated means are very close to the expected mean, and both simulated standard deviations are close to the expected standard deviation, being the first one almost equal.

TI-Nspire



1 Enter **randNorm**(40,2.5,50)→*samp_a* to store the 50 random numbers as *samp_a*

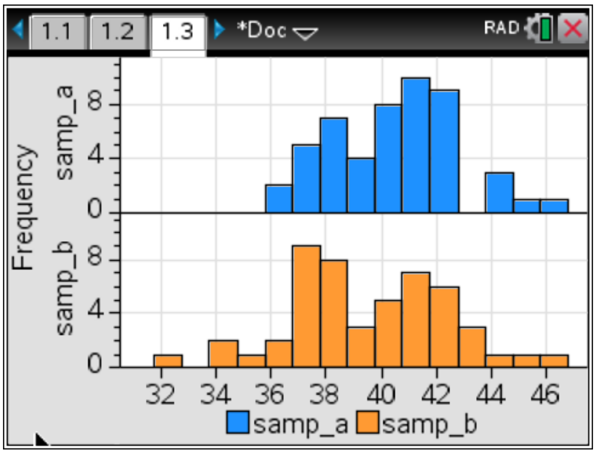
2 Enter **randNorm**(40,2.5,50)→*samp_b* to store the 50 random numbers as *samp_b*.



3 Add a **Lists & Spreadsheet** page.

4 Tap in the cell next to the **A**, press **var** and select *samp_a*.

5 Tap in the cell next to the **B**, press **var** and select *samp_b*.



6 Add a **Data & Statistics** page.

7 For the horizontal axis, select *samp_a*.

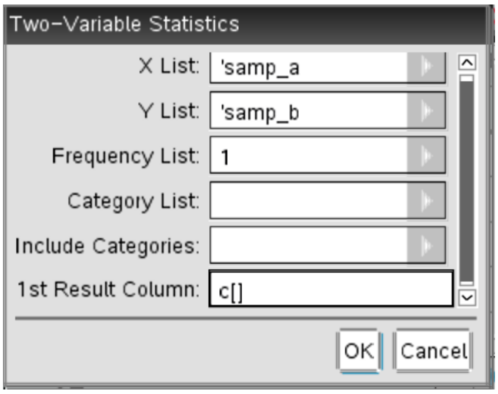
8 Press **menu** > **Plot Type** > **Histogram**.

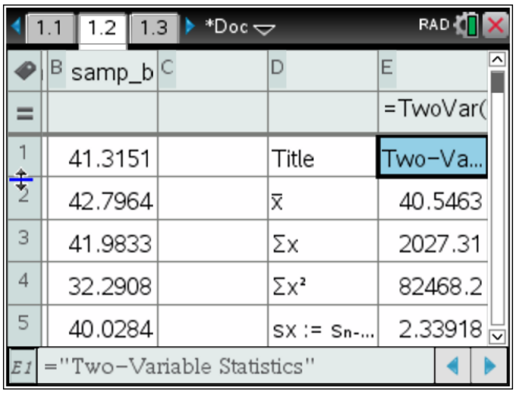
9 The sample a data will be displayed as a histogram.

10 Tap **menu** > **Plot Properties** > **Add X Variable**.

11 Select *samp_b*.

12 The data for both samples will be displayed as histograms.





13 Return to the **Lists & Spreadsheet page**.

14 Press **menu** > **Statistics** > **Stat Calculations** >**Two-Variable**.

15 In the **X List:** field, press the right arrow and select *samp_a*.

16 In the **Y List:** field, press the right arrow and select *samp_b*.

17 Press enter. The results are in columns **C** and **D**.

$\mathrm{E}(X) = 40,\ \mathrm{SD}(X) = 2.5$

E(sample 1) ≈ 40.5, SD(sample 1) ≈ 2.3

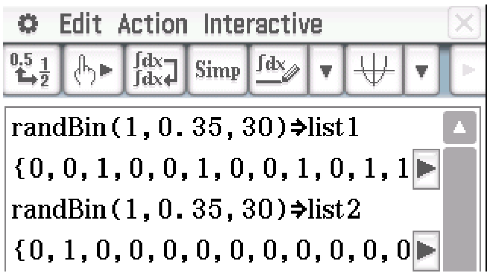
E(sample 2) ≈ 39.5, SD(sample 2) ≈ 2.9

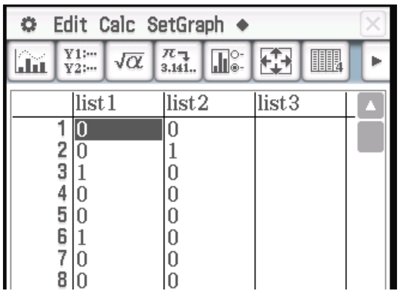
Both simulated means are quite close to the expected mean. However, only one of the simulated standard deviations is close to the expected value.

Question 6

Answers will vary depending on simulation.

ClassPad





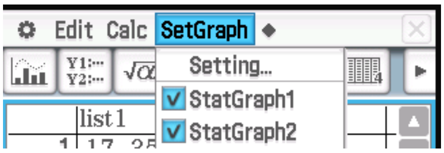
1 In **Catalog**, choose **randBin** (or type **randBin**).

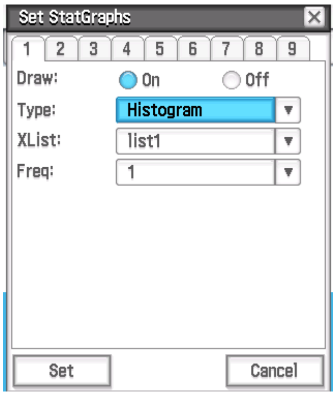
2 Enter and store the following formula in **list1**.

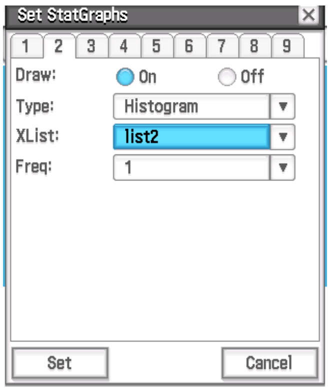
randBin(1,0.35,30)⇨**list1**

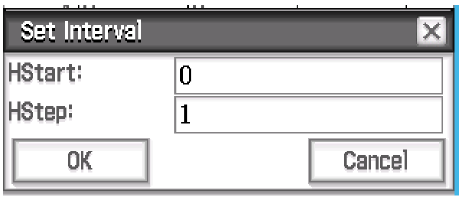
3 Repeat to obtain a second sample and store it in **list2**.

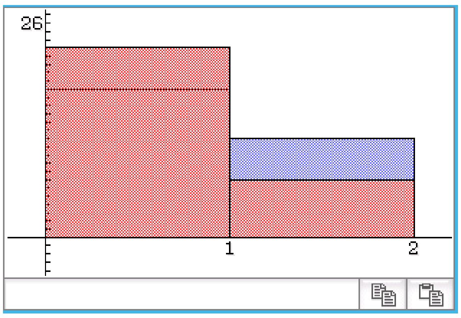
4 Tap **Menu** > **Statistics** to view the two lists.









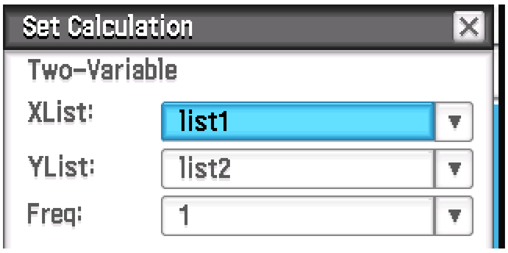


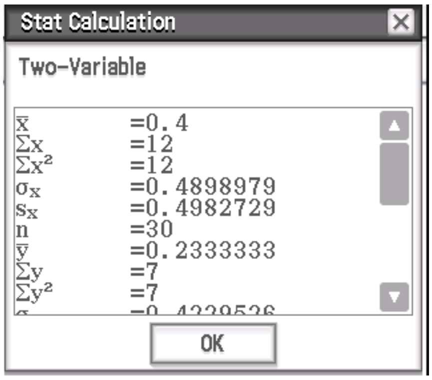
5 Tap **Setgraph** > **Setting**

6 Tap on the **1 tab** and change the **Type:** field to **Histogram** and the **XList** to **list1.**

7 Choose the 2 tab and change the **XList** to **list2.**

8 Tap Graph and change the **HStart:** field options to that shown.





9 Tap **Calc>Two-Variable** and choose **list1 and list2**

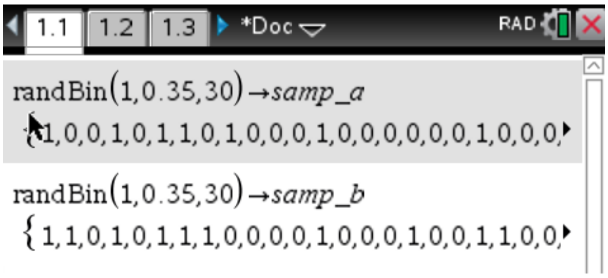
$\mathrm{E}(Z)=0.35$, $\mathrm{SD}(Z)=\sqrt{0.35(1-0.35)}\approx 0.48$

E(sample 1) = 0.4, SD(sample 1) ≈ 0.49

E(sample 2) ≈ 0.23, SD(sample 2) ≈ 0.43

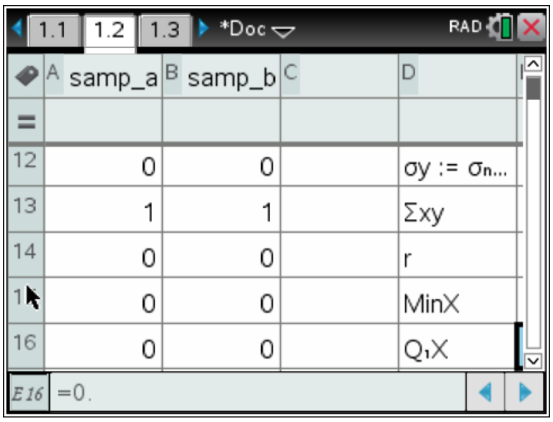
The first mean is close to the expected mean, and the first standard deviation is close to the expected standard deviation.

TI-Nspire



1 Enter **randBin**(1,0.35,30)→*samp_a* to store the 30 random numbers as *samp_a*

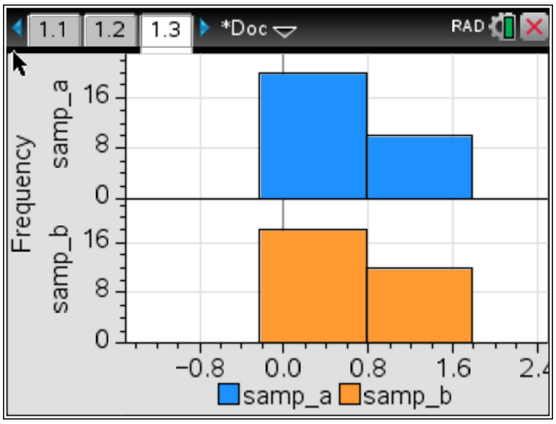
2 Enter **randBin**(1,0.35,30)→*samp_b* to store the 30 random numbers as *samp_b*.



3 Add a **Lists & Spreadsheet** page.

4 Tap in the cell next to the **A**, press **var** and select *samp_a*.

5 Tap in the cell next to the **B**, press **var** and select *samp_b*.



6 Add a **Data & Statistics** page.

7 For the horizontal axis, select *samp_a*.

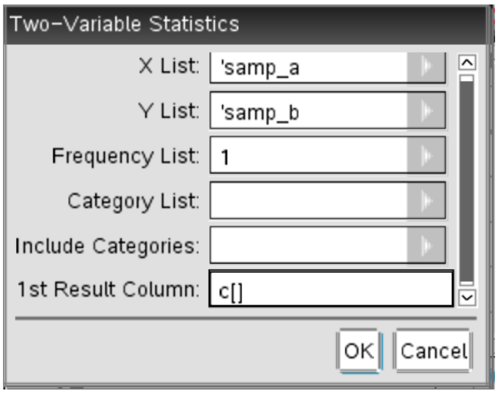
8 Press **menu** > **Plot Type** > **Histogram**.

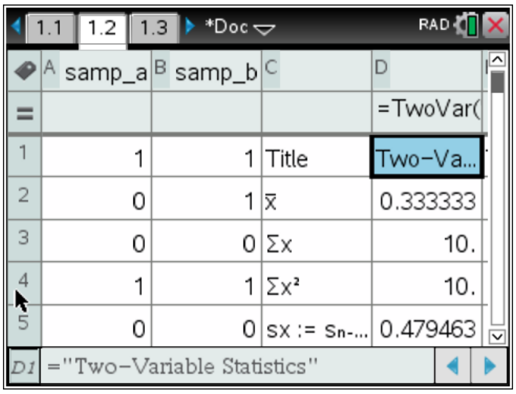
9 The sample a data will be displayed as a histogram.

10 Tap **menu** > **Plot Properties** > **Add X Variable**.

11 Select *samp_b*.

12 The data for both samples will be displayed as histograms.





13 Return to the **Lists & Spreadsheet page**.

14 Press **menu** > **Statistics** > **Stat Calculations** >**Two-Variable**.

15 In the **X List:** field, press the right arrow and select *samp_a*.

16 In the **Y List:** field, press the right arrow and select *samp_b*.

17 Press enter. The results are in columns **C** and **D**.

$$\text{E}(Z) = 0.35,\ \text{SD}(Z) = \sqrt{0.35(1-0.35)} \approx 0.48$$

E(sample 1) ≈ 0.33, SD(sample 1) ≈ 0.48

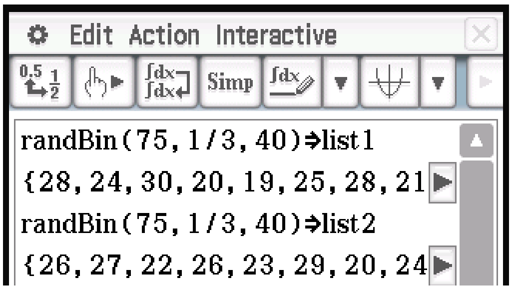
E(sample 2) = 4.0 , SD(sample 2) ≈ 0.5

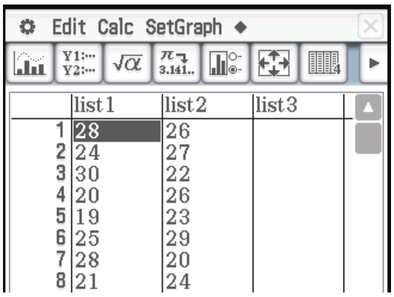
The first mean is close to the expected mean and the first simulated standard deviations is almost the same of the actual value.

Question 7

Answers will vary depending on simulation.

ClassPad





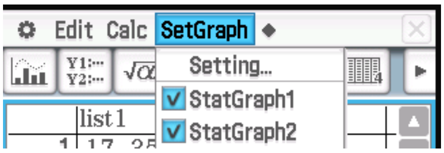
1 In **Catalog**, choose **randBin** (or type **randBin**).

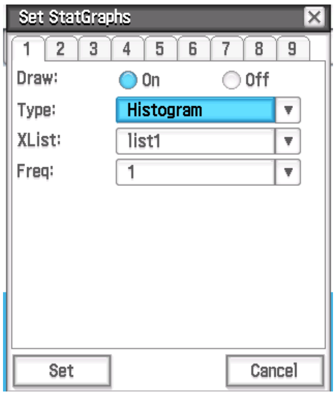
2 Enter and store the following formula in **list1**.

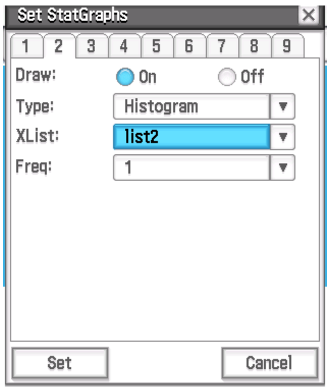
randBin(75,1/3,40)⇨**list1**

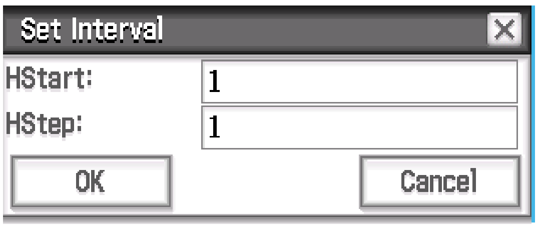
3 Repeat to obtain a second sample and store it in **list2**.

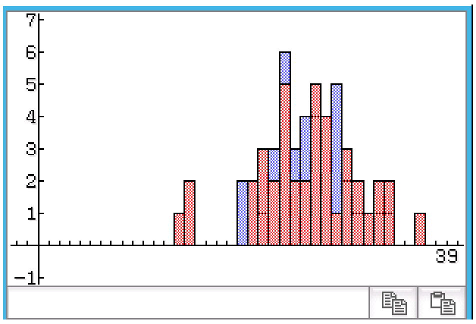
4 Tap **Menu** > **Statistics** to view the two lists.









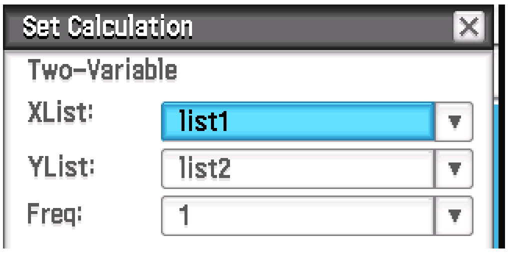


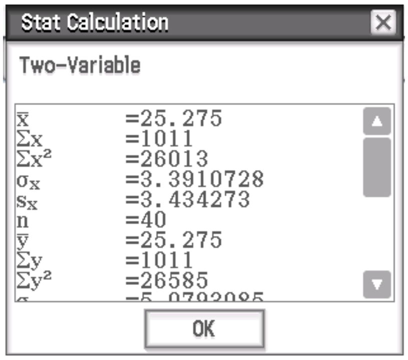
5 Tap **Setgraph** > **Setting**

6 Tap on the **1 tab** and change the **Type:** field to **Histogram** and the **XList** to **list1.**

7 Choose the 2 tab and change the **XList** to **list2.**

8 Tap Graph and change the **HStart:** field options to that shown.





9 Tap **Calc>Two-Variable** and choose **list1 and list2**

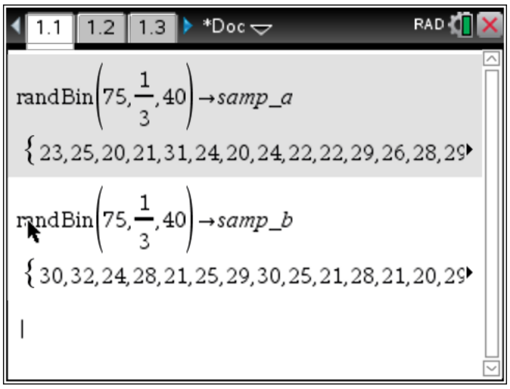
$$\text{E}(T) = 75\times\frac{1}{3} = 25\text{, SD}(T) = \sqrt{75\times\frac{1}{3}\left(1-\frac{1}{3}\right)} \approx 4.08$$

E(sample 1) = 25.275, SD(sample 1) ≈ 3.39

E(sample 2) = 25.275, SD(sample 2) ≈ 5.08

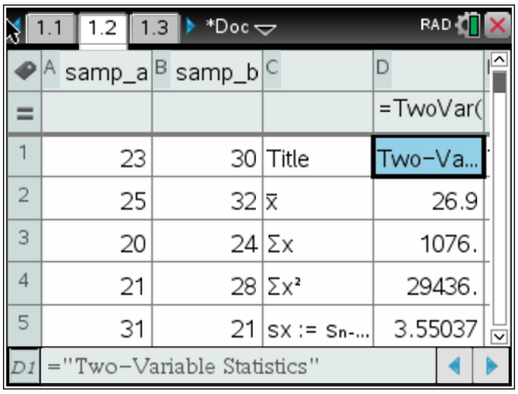
Both means are very close to the expected mean, but both standard deviations vary and are not close to the standard deviation.

TI-Nspire



1 Enter **randBin**(75,1/3,40)→*samp_a* to store the 30 random numbers as *samp_a*

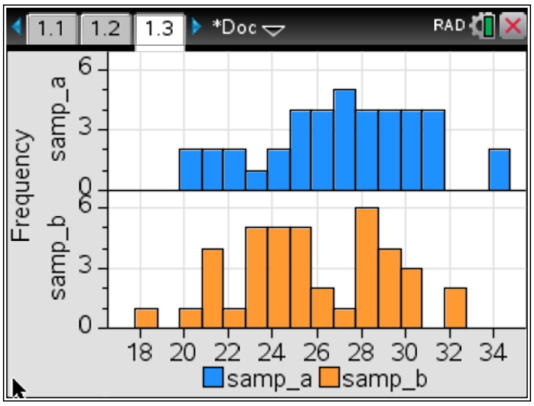
2 Enter **randBin**(75,1/3,40)→*samp_b* to store the 30 random numbers as *samp_b*.



3 Add a **Lists & Spreadsheet** page.

4 Tap in the cell next to the **A**, press **var** and select *samp_a*.

5 Tap in the cell next to the **B**, press **var** and select *samp_b*.



6 Add a **Data & Statistics** page.

7 For the horizontal axis, select *samp_a*.

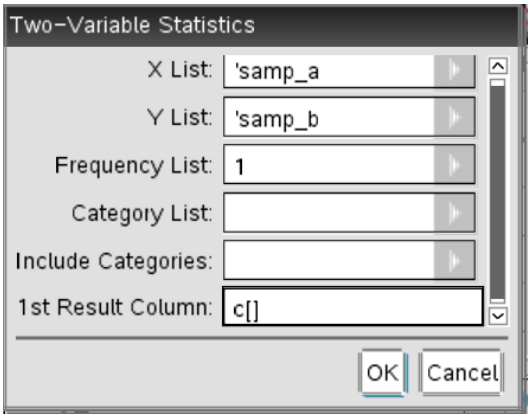
8 Press **menu** > **Plot Type** > **Histogram**.

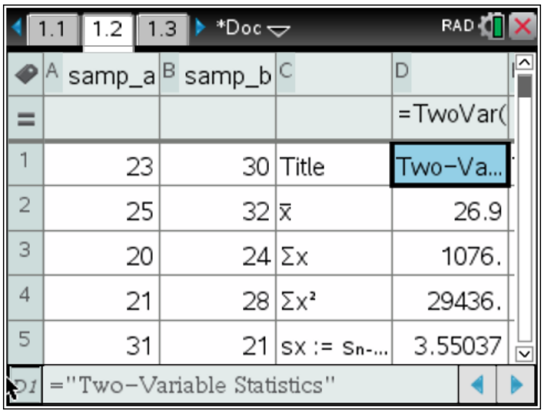
9 The sample a data will be displayed as a histogram.

10 Tap **menu** > **Plot Properties** > **Add X Variable**.

11 Select *samp_b*.

12 The data for both samples will be displayed as histograms.





13 Return to the **Lists & Spreadsheet page**.

14 Press **menu** > **Statistics** > **Stat Calculations** >**Two-Variable**.

15 In the **X List:** field, press the right arrow and select *samp_a*.

16 In the **Y List:** field, press the right arrow and select *samp_b*.

17 Press enter. The results are in columns **C** and **D**.

$$\mathrm{E}(T) = 75\times\frac{1}{3} = 25,\ \mathrm{SD}(T) = \sqrt{75\times\frac{1}{3}\left(1-\frac{1}{3}\right)} \approx 4.08$$

E(sample 1) ≈ 26.9, SD(sample 1) ≈ 3.55

E(sample 2) = 25.525, SD(sample 2) ≈ 3.43

The second mean is close to the expected mean and both simulated standard deviations are fairly near the standard deviation.

Question 8 [SCSA MM2017 Q12a] (4 marks)

(✓ = 1 mark)

a The method is biased due to:

- the people being asked a leading question
- the specific time and location used for the survey.

states method is biased with reason✓

states a correct reason✓

b In this case the question is not biased, however, only mobile phone users were selected causing bias. Also, many of these people may just hang up, causing non-response bias.

states the method is biased with reason✓

states a correct reason✓

Question 9 [SCSA MM2018 Q17c] (2 marks)

(✓ = 1 mark)

Tina could use a random number generator and pick the sample using the numbers she obtains.

Other answers are possible.

indicates some random mechanism✓

indicates that the mangoes are selected accordingly✓

Question 10 [SCSA MM2019 Q13bc] (4 marks)

(✓ = 1 mark)

a Any two of the following reasons:

1. Spatial: only one location, so only those present in that mall will be sampled from.
2. Temporal: lunchtime, so only those present at lunchtime will be sampled from.
3. Selection scheme: quota sampling means that the first 400 workers only are in the sample, so this is not a random sample from all workers.

States one source of bias with explanation✓

states a second source of bias with explanation✓

b Either of the following reasons:

1. Only those with listed telephone numbers will be selected.
2. Non-response bias: Not everyone will answer their phone when called.

one source of bias is outlined✓

c Amir's scheme is better, as it samples randomly from the whole population of workers.

states Amir's is the better sample with reason✓

Question 11 [SCSA MM2020 Q14c] (4 marks)

(✓ = 1 mark)

1. Temporal: The sample is at a fixed time, so only people around at that time will be sampled.
2. Spatial: The location is fixed, so:

   (i) only people at that location will be sampled **or**

   (ii) not everyone from the suburb will pass by that area;

   so this is not a random sample of the residents.

identifies one possible source of bias✓

explains why it is a possible source of bias✓

identifies another possible source of bias✓

explains why it is a possible source of bias✓

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Question 12 (9 marks)

(✓ = 1 mark)

a **The shapes of the distributions are seemingly different**✓, and **the ranges of values that the variable takes are not consistent**✓ in the samples (e.g., Sample A has scores between 8 and 18, Sample B has scores between 74 and 83 and Sample C has scores between 20 and 65.

b **i** **Sample A**.✓ A binomial distribution consists of discrete values and only two outcomes, success and failure.

ii The distribution appears positively skewed, so $\boldsymbol{p < 0.5}$.✓

c **i** **Sample B**.✓ The shape resembles the typical bell-shape.

ii **E(X) = 80**.✓ The distribution is approximately symmetrical about 80.

d **i** **Sample C**.✓ The shape is generally rectangular.

ii **The distribution of the sample should become more uniform; that is, there could be less variation in the heights of the columns and a 'flatter' distribution.**✓

EXERCISE 9.2 The sampling distribution of sample proportions

Question 1

a The population of interest is the **supporters of the local football team**.

b Some possible biases.

1. **Spatial:** only supporters at the game were asked. This does not make allowance for supporters that did not attend on that day.
2. **Spatial:** groups of people standing together were asked, meaning they could have travelled together using the same means of transport.
3. **Temporal**: data is only collected from one match. Different matches might attract different supporters who come to the game with a different means of transport.
4. **Self-selection bias**: Being close together, it is likely that all supporters will volunteer to answer regardless of whether or not they were asked.

Question 2

Parameters are characteristics of populations, while statistics are characteristics of samples.

That is, a statistic derives from a measurement.

A statistic is calculated from a sample and represents a characteristic of that sample, while a parameter represents a characteristic of the entire population from which the sample is drawn.

The answer is C.

Question 3

a $\hat{p} = \frac{x}{n}$

Calculate the proportion, with $x = 28$, $n = 156$.

$$\boldsymbol{\hat{p} = \frac{28}{156} \approx 0.1795 = 17.95\%}$$

b $0.1795 \times 980 = \mathbf{176}$ students

Question 4

a $\mathrm{E}(\hat{p}) = p = \mathbf{0.8}$

b $\mathrm{Var}(\hat{p}) = \dfrac{p(1-p)}{n} = \dfrac{0.8(1-0.8)}{125} = 0.00128 \approx \mathbf{0.0013}$

c $\mathrm{SD}(\hat{p}) = \sqrt{\mathrm{Var}(\hat{p})} = \sqrt{0.00128} \approx \mathbf{0.036}$

Question 5

a $\mathrm{E}(\hat{p}) = \dfrac{9}{200} = \mathbf{0.045 = 4.5\%}$

b **i** $\mathrm{E}(\hat{p}) = p = \mathbf{0.045}$

ii $\mathrm{Var}(\hat{p}) = \dfrac{p(1-p)}{n} = \dfrac{0.045(1-0.045)}{200} = 0.000214875 \approx \mathbf{0.0002}$

iii $\mathrm{SD}(\hat{p}) = \sqrt{\mathrm{Var}(\hat{p})} = \sqrt{0.000214875} \approx \mathbf{0.015}$

Question 6

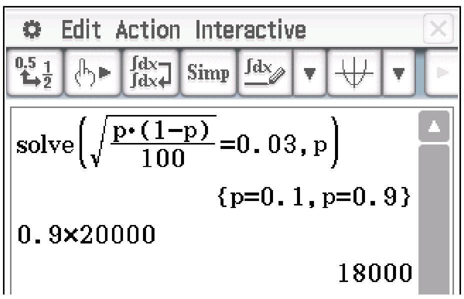
Use the standard deviation formula $\mathrm{SD}(\hat{p}) = \sqrt{\dfrac{p(1-p)}{n}}$.

The standard deviation is 0.03 and the number of samples is $n = 100$.

Solve to determine the probability p. As there are more blue marbles than red marbles, the probability of selecting a blue marble is 0.9.

The number of blue marbles in the box is 0.9 × 20 000 = 18 000.

ClassPad



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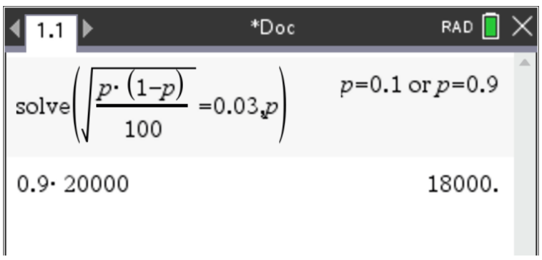
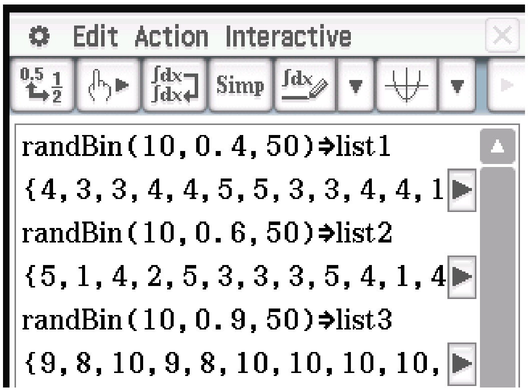


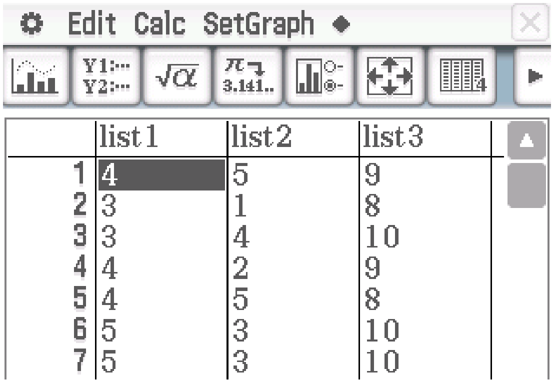
The number of blue marbles is 18 000.

Question 7

Answers will vary depending on simulation.

ClassPad





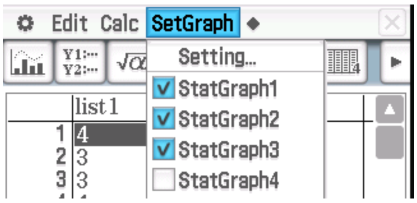
1 In **Catalog**, choose **randBin** (or type **randBin**).

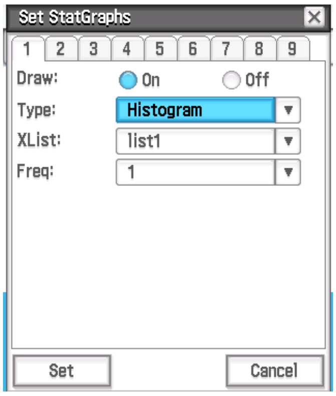
2 Enter **randBin**(10,0.4,50)⇨**list1to store the first sample in list1.**

3 Enter **randBin**(10,0.6,50)⇨**list1to store the first sample in list2.**

4 Enter **randBin**(10,0.9,50)⇨**list1to store the first sample in list3.**

5 Tap **Menu** > **Statistics** to view the two lists.





5 Tap **Setgraph** > **Setting**

6 Tap on the **1 tab** and change the **Type:** field to **Histogram** and the **XList** to **list1.**

7 Choose the **2 tab** and change the **XList** to **list2.**

8 Choose the **3 tab** and change the **XList** to **list3.**

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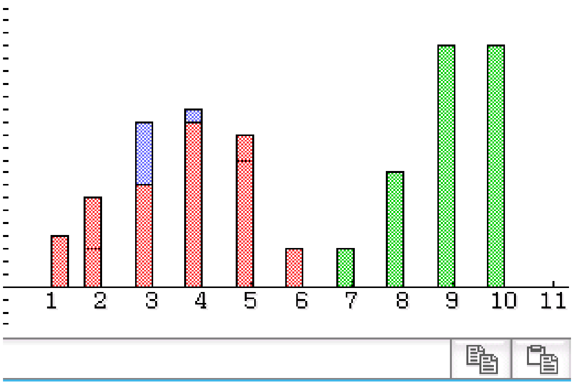
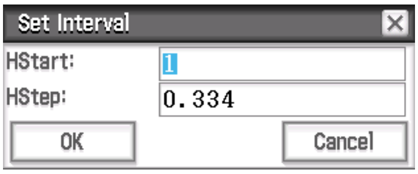
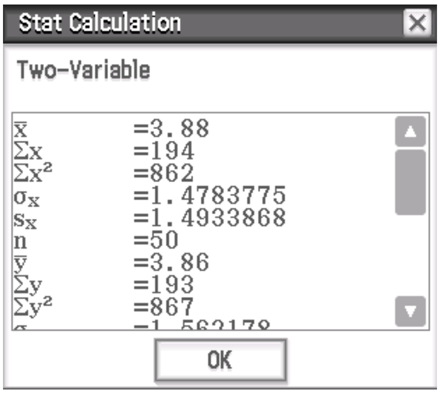




9 Tap Graph and leave the **HStart:** field options to the default values.

Histograms of the three different samples will be displayed.



10 Tap **Calc>One-variable** and choose **XList: list1** and write down the mean**.**

11 Tap **Calc>One-variable** and choose **XList: list2** and write down the mean**.**

12 Tap **Calc>One-variable** and choose **XList: list3** and write down the mean**.**

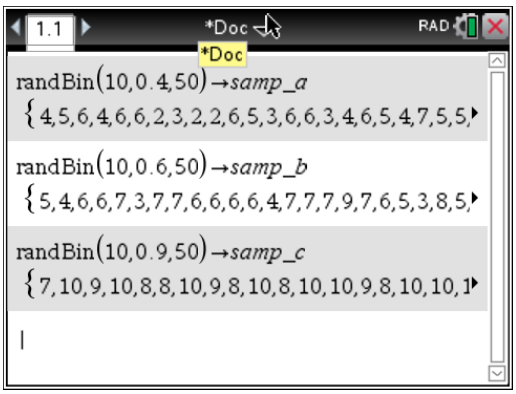
The mean of the first sample is 3.88 (expected $10 \times 0.4 = 4$).

The mean of the second sample is 3.86 (expected $10 \times 0.6 = 6$).

The mean of the third sample is 9.08 (expected $10 \times 0.9 = 9$).

All three graphs display a slight negative skewness.

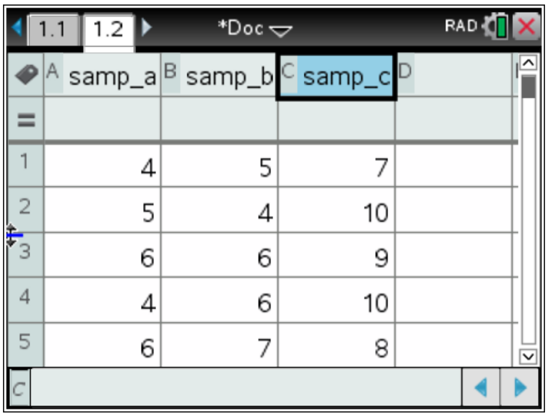
TI-Nspire



1 Enter **randBin**(10,0.4,50)→*samp_a* to store 50 numbers as *samp_a*

2 Enter **randBin**(10,0.6,50)→*samp_b* to store 50 numbers as *samp_b*.

3 Enter **randBin**(10,0.9,50)→*samp_c* to store 50 numbers as *samp_c*.

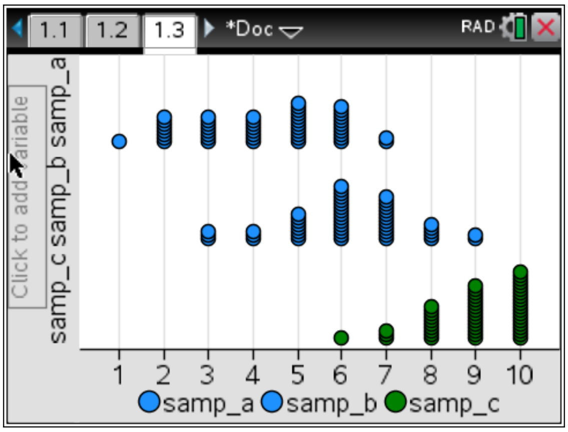


4 Add a **Lists & Spreadsheet** page.

5 Tap in the cell next to the **A**, press **var** and select *samp_a*.

6 Tap in the cell next to the **B**, press **var** and select *samp_b*.

7 Tap in the cell next to the **C**, press **var** and select *samp_c*.



6 Add a **Data & Statistics** page.

7 For the horizontal axis, select *samp_a*.

8 Press **menu** > **Plot Type** > **Histogram**.

9 The sample a data will be displayed as a histogram.

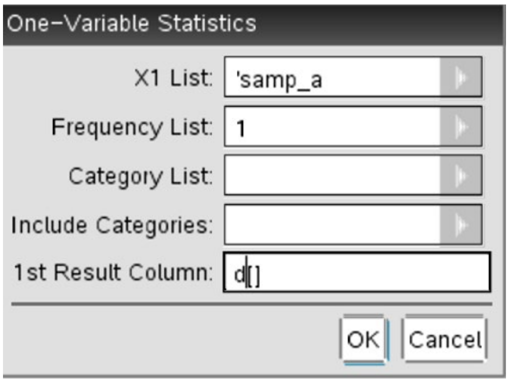
10 Tap **menu** > **Plot Properties** > **Add X Variable**.

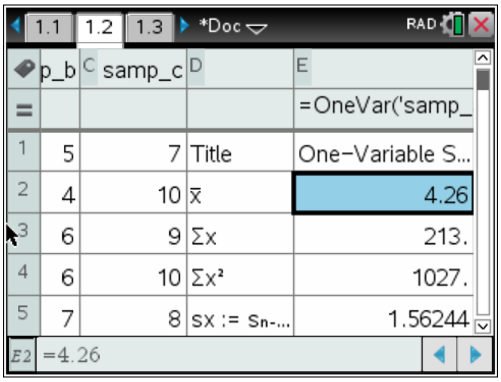
11 Select *samp_b*.

12 Tap **menu** > **Plot Properties** > **Add X Variable**.

13 Select *samp_c*.

14 The data for the three samples will be displayed as histograms.





13 Return to the **Lists & Spreadsheet page**.

14 Press **menu** > **Statistics** > **Stat Calculations** >**One-Variable** and choose the options shown to find the mean for sample_a. Record the result.

15 Press **menu** > **Statistics** > **Stat Calculations** >**One-Variable** and choose the options shown to find the mean for sample_b. Record the result.

16 Press **menu** > **Statistics** > **Stat Calculations** >**One-Variable** and choose the options shown to find the mean for sample_c. Record the result.

E(sample_a) = 4.26

E(sample_b) = 6.12

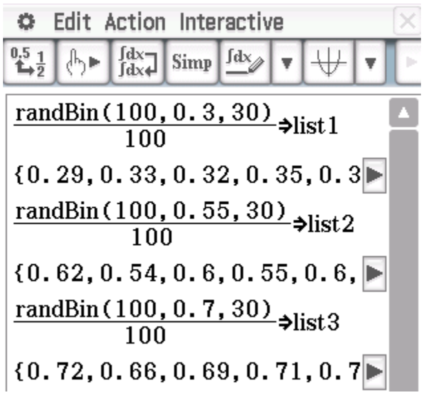
E(sample_c) = 9.02

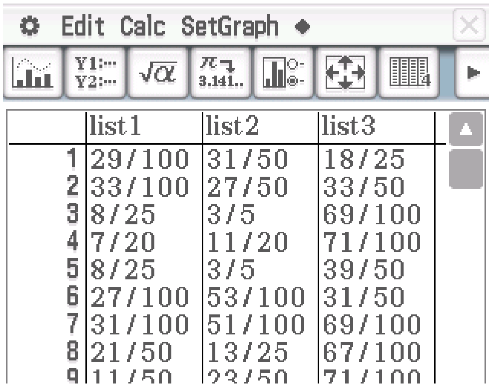
Sample_a and sample_c are slightly negatively skewed.

Question 8

Answers will vary depending on simulation.

ClassPad





1 Open the **Keyboard** then tap **Catalog** > **R** to jump to the functions starting with r.

2 Select **randBin(**.

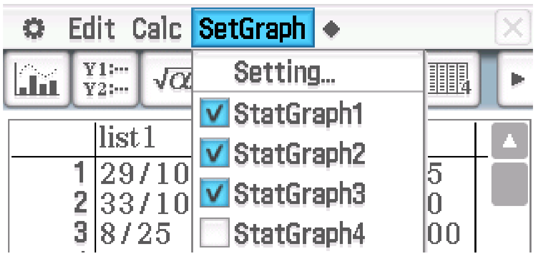
3 Generate 30 observations of a binomial random variable, with $n = 100$ trials and $p = 0.3$, and divide the set by 100.

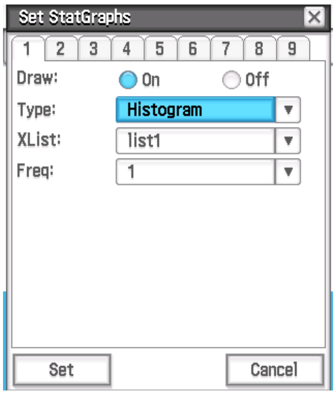
4 Store the values into **list1**.

5 Repeat for the second and third set of values, storing the results into **list2** and **list3** respectively.

6 Tap **Menu** > **Statistics**.

7 The values generated will be displayed in **list1**, **list2** and **list3**.



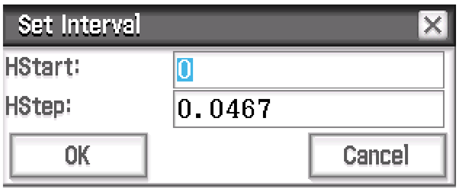


5 Tap **Setgraph > Setting**

6 Tap on the **1 tab** and change the **Type:** field to **Histogram** and the **XList** to **list1.**

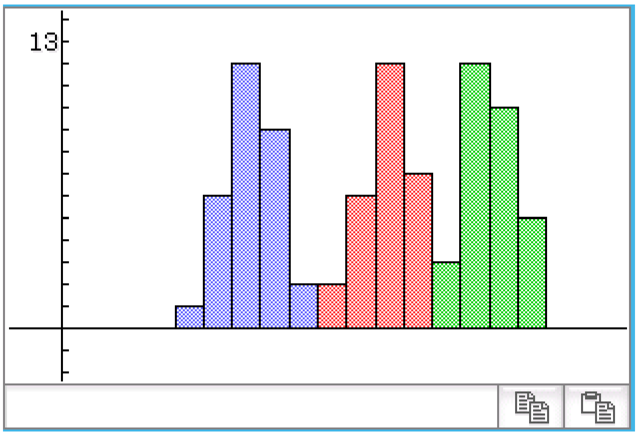
7 Choose the **2 tab** and change the **XList** to **list2.**

8 Choose the **3 tab** and change the **XList** to **list3.**

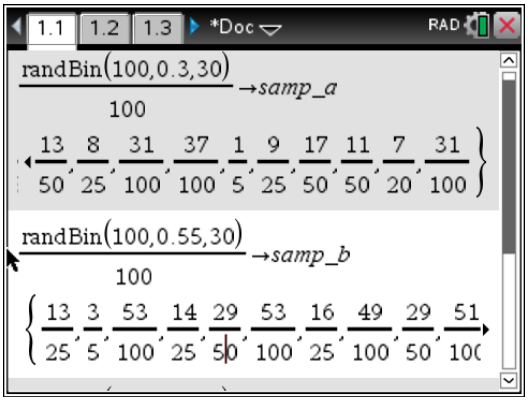


9 Tap Graph and leave the **HStart:** field options to the default values.

Histograms of the three different samples will be displayed.



TI-Nspire

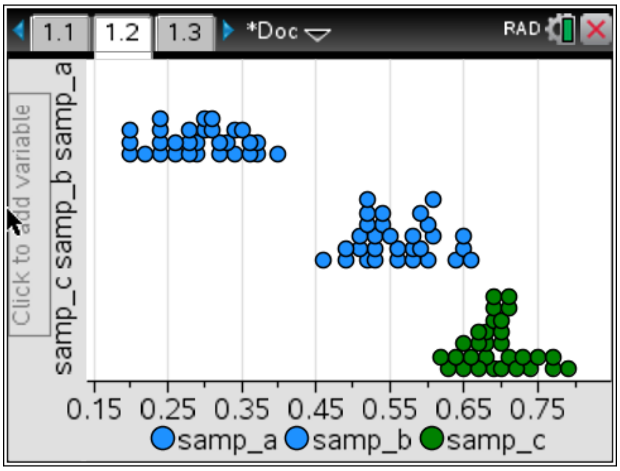


1 Press **menu** > **Probability** > **Random** > **Binomial**.

2 Generate 30 observations of a binomial random variable, with n = 100 trials and p = 0.3, and divide the set by 100.

3 Press **ctrl** > **var** to store the result in *samp_a*.

4 Repeat for the second and third set of values, storing the results in samp_b and *samp_c* respectively.



6 Add a **Data & Statistics** page.

7 For the horizontal axis, select *samp_a*.

8 The sample data will be displayed as a dot plot.

9 Tap **menu** > **Plot Properties** > **Add X Variable**.

10 Select *samp_b*.

11 Tap **menu** > **Plot Properties** > **Add X Variable**.

12 Select *samp_c*.

13 The data for the three samples will be displayed as dot plots.

Question 9

a Calculate the proportion, with $x = 10$, $n = 22$.

$$\hat{p} = \frac{x}{n} = \frac{10}{22} = \mathbf{0.4545}$$

b Calculate the proportion, with $x = 10$, $n = 22$.

$$\text{E}(\hat{p}) = p = \mathbf{0.4545}$$

$$\text{SD}(\hat{p}) = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.4545(1-0.4545)}{22}} = 0.1062$$

c $\hat{p} \approx 0.5$, hence the distribution is approximately symmetrical and since there are at least 10 each of positive and negative observations, the distribution is also approximately normal with $\hat{p} \sim N(0.4545.0.1062^2)$ (4 d.p.).

Question 10

a $\hat{p} = \dfrac{122}{500} = 0.244$

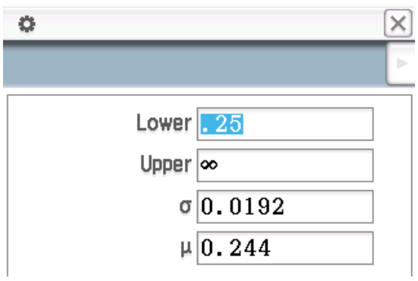
$np = 122 \geq 10$ and $n(1-p) = 500 \times (1-0.244) = 378 \geq 10$.

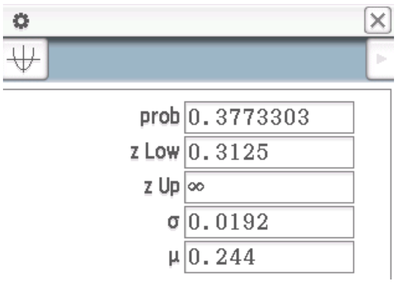
$\text{SD}(\hat{p}) = \sqrt{\dfrac{0.244(1-0.244)}{500}} = 0.0192$

Hence $\hat{p}$ is approximately normal, with $\hat{p} \sim N(0.244, 0.0192^2)$

b **i** Calculate $\text{P}(p > 0.25)$

ClassPad





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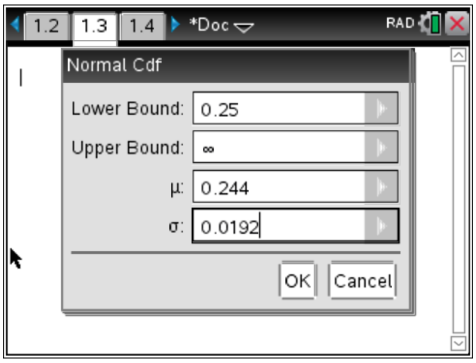
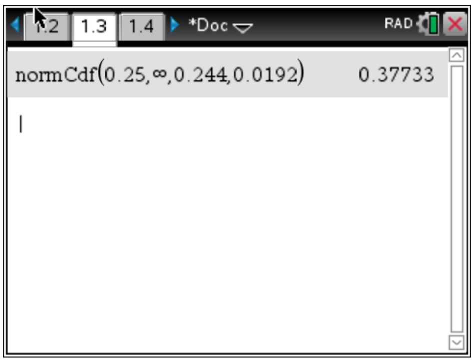
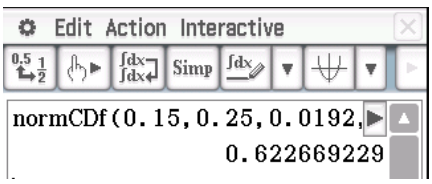




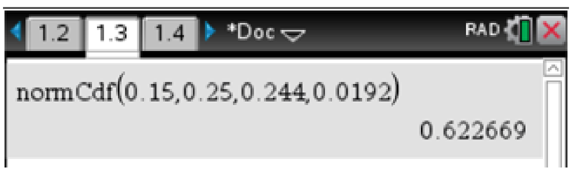
$\text{P}(p > 0.25) = 0.3774$

ii Calculate $\text{P}(0.15 \leq p \leq 25)$

ClassPad



TI-Nspire



$\text{P}(0.15 \leq p \leq 25) = \mathbf{0.6227}$

Question 11

a Calculate the proportion, with $x = 8$, $n = 125$.

$$\hat{p} = \frac{x}{n} = \frac{8}{125} = 0.064 = 6.4\%$$

The sample proportion of non-sellable items is **0.064**.

b $np = 125 \times 0.64 = 8$. Hence, although n is sufficiently large, $np < 10$, so that the distribution might not be symmetrical as is required for the normal model.

Also, $\text{SD}(\hat{p}) = \sqrt{\dfrac{0.064(1-0.064)}{125}} = 0.02189$ and $3 \times \text{SD}(\hat{p}) = 0.0657$.

Hence, $0.064 - 3 \times \text{SD}(\hat{p}) < 0$, meaning a normal distribution is not appropriate.

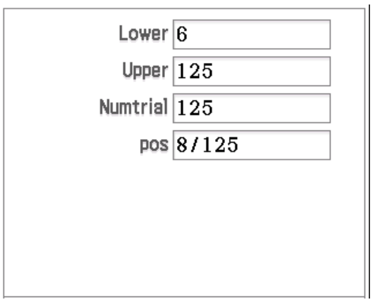
c Use a binomial distribution.

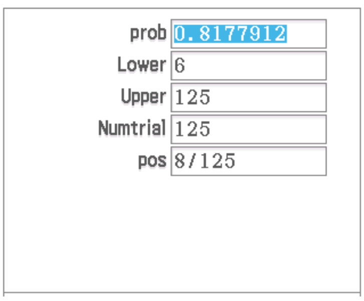
Let X be the number of non-sellable items.

Then we have $X \sim \text{Bin}\left(125, \dfrac{8}{125}\right)$.

Calculate $\text{P}(X \geq 6)$

ClassPad





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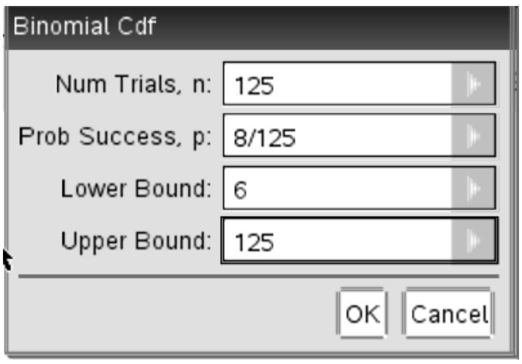
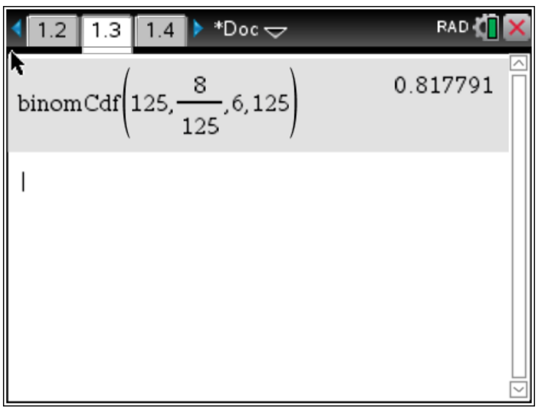
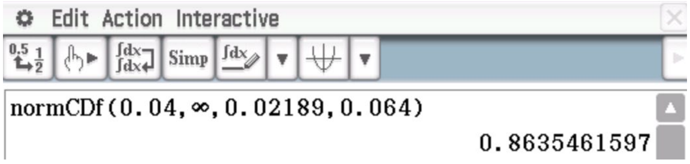




$\text{P}(X \geq 6) = \mathbf{0.8178}$

d Find the normal approximation using $\frac{5}{125} = 0.04$, mean 0.064 and standard deviation 0.02189.

ClassPad



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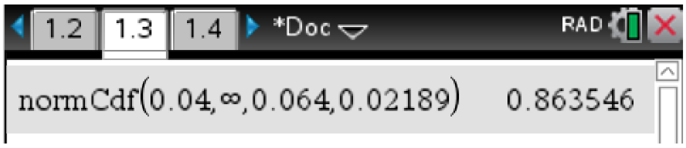

Hence with a normal distribution $P(X \geq 5) = 0.8635$

With a binomial distribution $P(X > 5) = 0.8178$

There is approximately 0.05 difference between the two results.

Question 12

a $p = 0.41$ is close to 0.5, both $np = 49.2$ and $n(1-p) = 70.8$ are greater than 10.

Also, $n = 140$ is sufficiently large to justify that the distribution will be fairly symmetrical and thus that an approximate normal distribution will be appropriate.

b

$$\text{SD}(\hat{p}) = \sqrt{\frac{0.41(1-0.41)}{120}} = 0.04490$$

$$z = \frac{0.5-0.41}{0.04490} = 2.004$$

This is approximately 2 standard deviations above the mean.

Question 13

a $$\text{Var}(\hat{p}) = \frac{\hat{p}(1-\hat{p})}{n}$$

$$0.00308 = \frac{0.56(1-0.56)}{n}$$

Solving for n gives $n = 80$

The sample size is 80.

b Convert the interval $P(0.5 < \hat{p} < 0.62)$ to standard score.

$$2k = \frac{0.62 - 0.56}{\sqrt{0.00308}} = 2.1622$$

$k =$ **1.0811**

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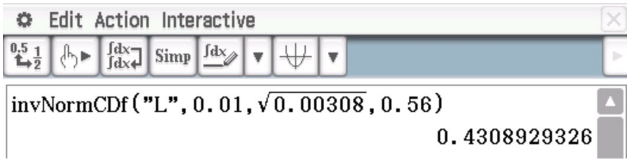
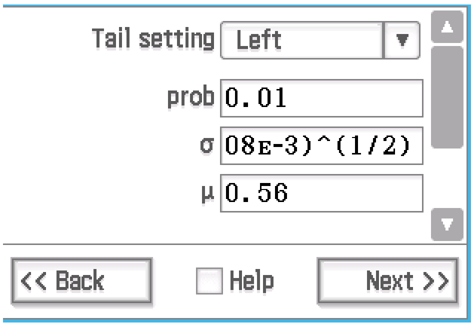
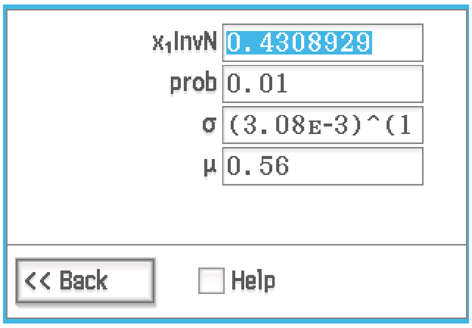
c Use **invNORM**

ClassPad



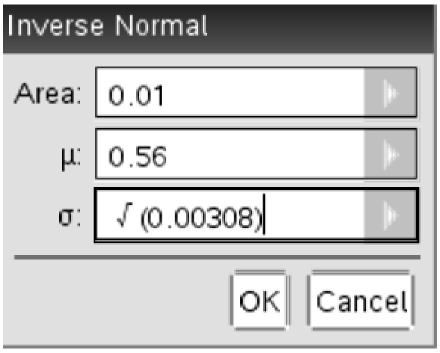
or





$\hat{p} = 0.43$

TI-Nspire



Question 14 (4 marks)

(✓ = 1 mark)

a $\hat{p} = 0.12$

$0.12 \times 75 =$ **9 households have more than three bedrooms.**✓

b $\boldsymbol{np = 9 < 10}$ ✓, so the distribution may not be sufficiently close to being symmetrical to allow a normal distribution.

c The values (number of bedrooms) are discrete, so a binomial distribution will apply.

$\mathrm{P}(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$, where from n households, x households have more than three bedrooms.

For exactly n = 10 households, use p = 0.12 and x = 2.✓

Hence $\mathbf{P}(X = 2) = \binom{10}{2} 0.12^2 (0.88)^8$ ✓

Question 15 (3 marks)

(✓ = 1 mark)

The values (number of people) is discrete, so we use a binomial distribution.

$\mathrm{P}(\hat{p}=0)$ means the probability of success (proportion who live in a capital city) is zero.✓

Hence $X = 0$ so from $\mathrm{P}(X=x)=\binom{n}{x}p^x(1-p)^{n-x}$ we have

$$\mathbf{P}(X=0)=\binom{5}{0}p^0(1-p)^5=(1-p)^5 \checkmark$$

$$\mathrm{P}(\hat{p}=0)=\frac{1}{243}$$

$$(1-p)^5=\frac{1}{3^5}$$

$$1-p=\left(\frac{1}{3^5}\right)^{\frac{1}{5}}=\frac{1}{3}$$

Hence $\boldsymbol{p=1-\frac{1}{3}=\frac{2}{3}}$ ✓

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Question 16 (7 marks)

(✓ = 1 mark)

a

$$\text{Var}(\hat{p}) = \frac{\hat{p}(1-\hat{p})}{n} \checkmark$$

$$0.08^2 = \frac{0.2(1-0.2)}{n} \checkmark$$

Solve for n.

$n = 25$✓

b

$$\text{Var}(\hat{p}) = \frac{\hat{p}(1-\hat{p})}{n}$$

$$0.04^2 = \frac{p(1-p)}{100} \checkmark$$

Solve for p using CAS or algebra.

$$16 = 100p - 100p^2$$
$$25p^2 - 25p + 4 = 0$$

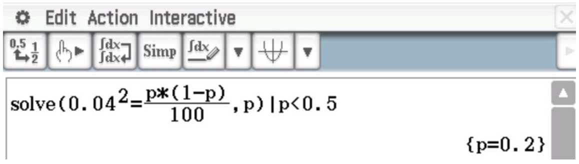
$$p = \frac{25 \pm \sqrt{625-400}}{50}$$
$$= \frac{25 \pm \sqrt{225}}{50} = \frac{25 \pm 15}{50} \checkmark$$

$$p = \frac{1}{5},\ p = \frac{4}{5} \checkmark$$

Since $p < 0.5$, take $p = \frac{1}{5}$ ✓

Or

ClassPad



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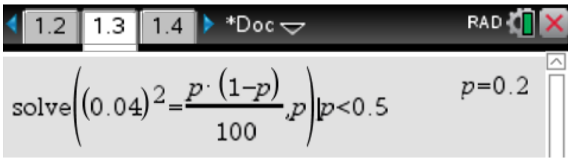

correct formula used✓

correct value used for variance✓

restriction placed on the value of *p*✓

correct answer for *p*✓

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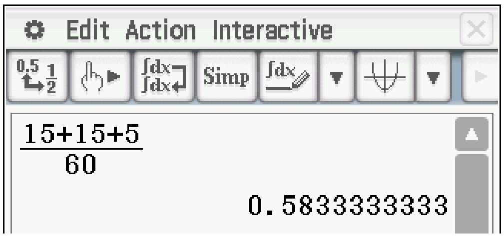
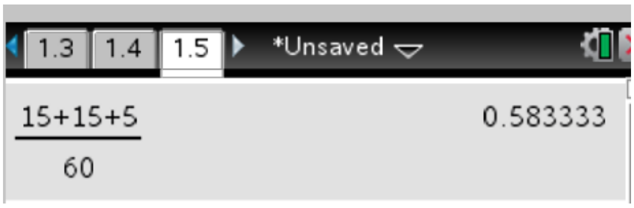
Question 17 [SCSA MM2016 Q14abd] (6 marks)

(✓ = 1 mark)

a **ClassPad** **TI-Nspire**





identifies the prime numbers✓

determines proportion✓

b $\hat{p} = 0.58$

$$s_x = \sqrt{\frac{0.58(1-0.58)}{60}} = 0.0636\ldots$$

determines the mean✓

determines standard deviation✓

c Graph takes the shape of a binomial distribution.

Approaches the shape of a normal distribution for large values of n.

Distribution is centred on 0.58.

at least one of the descriptors above✓

at least two descriptors above✓

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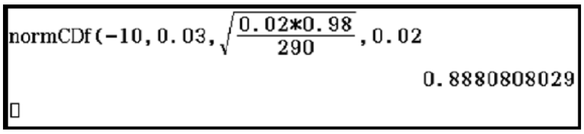
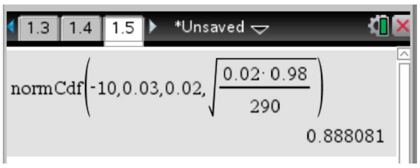
Question 18 [SCSA MM2017 Q18d MODIFIED] (6 marks)

(✓ = 1 mark)

a **ClassPad** **TI-Nspire**





i.e. a probability of approximately 0.89.

CDF up to 0.03✓

Determines standard deviation✓

calculates probability✓

b $np = 290 \times 0.02 = 5.8$. Although n is sufficiently large, $np < 10$, so that the distribution might not be symmetrical as is required for the normal distribution model.

Also, $\text{SD}(p) = \sqrt{\dfrac{0.02(1-0.02)}{290}} = 0.00822$ and $3 \times \text{SD}(p) = 0.02466$.

So $p - 3 \times \text{SD}(p) = 0.02 - 0.02466 < 0$, $p + 3 \times \text{SD}(p) = 0.02 + 0.02466 < 1$

use either $np = 5.8$ or standard deviation as explanation✓

c A larger sample will result in a smaller standard deviation, which means the distribution will be narrower. This will mean a greater area under the curve corresponding to $\hat{p} \le 0.03$, so the probability will increase.

states either that standard deviation will decrease as sample size increases, or the distribution will become narrower✓

states that probability will increase increases✓

Question 19 [SCSA MM2018 Q17abd] (6 marks)

(✓ = 1 mark)

a $\hat{p} \sim N\left(0.6, \dfrac{0.6 \times 0.4}{500}\right)$

□□□□□□□□□

$\hat{p} \sim N\left(0.6, 0.02191^2\right)$

states the distribution as normal✓

gives the correct value of the mean✓

gives the correct value of the variance (or standard deviation)✓

b

$$P\left(\hat{p} < 0.58\right) = P\left(Z < \dfrac{0.58 - 0.6}{\sqrt{\dfrac{0.6 \times 0.4}{500}}}\right) = P(Z < -0.9129) = 0.18065$$

calculates the *z*-value correctly✓

obtains the correct probability✓

c $\hat{p} = \dfrac{250}{500} = 0.5$

calculates the correct sample proportion✓

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Question 20 [SCSA MM2019 Q13a] (4 marks)

(✓ = 1 mark)

a $\hat{p} \sim N\left(0.4, \dfrac{0.4 \times 0.6}{400}\right)$

That is,

$\hat{p} \sim N(0.4, 0.0006)$

states normal distribution with correct mean✓

gives correct value of variance of standard deviation✓

b $P(\hat{p} > 0.44) = P\left(Z > \dfrac{0.44 - 0.4}{\sqrt{0.0006}}\right) = P(Z > 1.6330) = 1 - 0.9488$

$= 0.0512$

uses distribution from part a✓

determines correct probability✓

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Question 21 [SCSA MM2021 Q10acd MODIFIED] (6 marks)

(✓ = 1 mark)

a $\hat{p} = \dfrac{124}{400} = 0.31$

correctly determines the sample proportion✓

b $\text{mean} = p = \dfrac{7}{24} = 0.2917$

$$\text{standard deviation} = \sqrt{\dfrac{\frac{7}{24}\left(1 - \frac{7}{24}\right)}{400}} = 0.02273$$

correctly determines the mean✓

correctly determines the standard deviation✓

c $\left|\text{Second } \hat{p} - p\right| = 0.6 \times 0.02273 = 0.01364$

$\left|\text{Second } \hat{p}\right| = 0.2917 \pm 0.01364$

Possible number of prizes are: $400 \times (0.2917 \pm 0.01364) \approx 111 \text{ or } 122$

correctly determines the difference between the sample and population means✓

states the two possibilities for second event sample proportion✓

determines the possible number of prizes✓

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EXERCISE 9.3 Confidence intervals for proportions

Question 1

For **A**, $np = 16 \geq 10$ and $n(1-p) = 4 < 10$

For **B**, $np = 9 < 10$ and $n(1-p) = 6 < 10$

For **C**, $np = 10 \geq 10$ and $n(1-p) = 30 \geq 10$

For **D**, $np = 20 \geq 10$ and $n(1-p) = 20 \geq 10$

For **E**, $np = 0.9 < 10$ and $n(1-p) = 89.1 \geq 10$

Options C and D have $np \geq 10$ and $n(1-p) \geq 10$. However, the probability for D is $p = 0.5$, which is a better value than $p = 0.25$ for C in terms of a normal distribution.

The correct response is **D**.

Question 2

$\hat{p} = 0.09$

Write the formula for SD.

$$\text{SD}(\hat{p}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Substitute and solve n.

$$\sqrt{\frac{0.09(1-0.09)}{n}} = 0.03$$

$$\frac{0.09 \times 0.91}{n} = 0.03^2$$

$$n = \frac{0.09 \times 0.91}{0.03^2} = 91$$

The sample size was approximately 90.

Therefore, the correct response is **D**.

Question 3

a Estimate the value of p.

$\hat{p}=\dfrac{25}{50}=0.5\sim p$ so the distribution will be approximately symmetrical. Also, $np=25\geq 10$ and $n(1-p)=25\geq 10$, so a normal distribution is justified.

$$\mathrm{E}(\hat{p})=0.5$$

$$\mathrm{SD}(\hat{p})=\sqrt{\frac{0.5(0.5)}{50}}=0.07071$$

$$\hat{p}\sim N\left(0.5,0.0707^2\right)$$

b **i** Use $\left[\hat{p}-z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}},\ \hat{p}+z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right]$ with $\hat{p}=0.5,\ z=1.645,\ n=50$

$$\left[0.5-1.645\sqrt{\frac{0.5(1-0.5)}{50}},\ 0.5+1.645\sqrt{\frac{0.5(1-0.5)}{50}}\right]$$

or

$$0.5-1.645\sqrt{\frac{0.5^2}{50}}\leq p\leq 0.5+1.645\sqrt{\frac{0.5^2}{50}}$$

ii Use $\hat{p}=0.5,\ z=1.960,\ n=50$

$$\left[0.5-1.960\sqrt{\frac{0.5(1-0.5)}{50}},\ 0.5+1.960\sqrt{\frac{0.5(1-0.5)}{50}}\right]$$

or

$$0.5-1.960\sqrt{\frac{0.5^2}{50}}\leq p\leq 0.5+1.960\sqrt{\frac{0.52^2}{50}}$$

iii Use $\hat{p}=0.5,\ z=2.576,\ n=50$

$$\left[0.5-2.576\sqrt{\frac{0.5(1-0.5)}{50}},\ 0.5+2.576\sqrt{\frac{0.5(1-0.5)}{50}}\right]$$

or

$$0.5-2.576\sqrt{\frac{0.5^2}{50}}\leq p\leq 0.5+2.576\sqrt{\frac{0.5^2}{50}}$$

Question 4

Find the sample proportion.

$$\hat{p} = \frac{95}{140} \approx 0.6786$$

State the value of $z_{0.90} \approx 1.645$.

Use the formula for the confidence interval.

$$\text{Confidence interval} \approx \left[\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right]$$

Substitute in the values.

$$\left[0.6786 - 1.645\sqrt{\frac{0.6786(1-0.6786)}{140}}, 0.6786 + 1.645\sqrt{\frac{0.6786(1-0.6786)}{140}} \right]$$

$$\approx [0.614, 0.744]$$

The 90% confidence interval for songs on ‘drive time’ music radio being less than 3 minutes long is about **[0.614, 0.743]**.

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Question 5

a Find the sample proportion.

$\hat{p} = \frac{110}{300} \approx 0.36667$

State the value of $z_{0.95} \approx 1.960$.

$$\text{Confidence interval} \approx \left[\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right]$$

$$\left[0.36667 - 1.960\sqrt{\frac{0.36667(1-0.36667)}{300}},\ 0.36667 + 1.960\sqrt{\frac{0.36667(1-0.36667)}{300}}\right]$$

$\approx$ **[0.312, 0.421]**

b **Neither one is more likely than the other to contain *p*, as once observed, the probability that a confidence interval contains *p* is either 0 or 1. Hence, it cannot be determined.**

c In a 95% confidence interval, it is expected that approximately 95% of the intervals constructed from random samples will contain the true value of the population parameter, which is the proportion p.

Hence we have $0.95 \times 240 =$ 228 confidence intervals

Question 6

a

Let the confidence interval be $[\hat{p} - a, \hat{p} + a] = \left[\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right]$

If z is reduced, then a will reduce. Hence $\hat{p} - a$ will decrease and $\hat{p} + a$ will decrease. Thus **the width of the confidence interval will decrease.**

b

Let the confidence interval be $[\hat{p} - a, \hat{p} + a] = \left[\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right]$

If n is increased, a will decrease.

Hence $\hat{p} - a$ will decrease and $\hat{p} + a$ will decrease. Thus **the width of the confidence interval will decrease.**

Question 7

a

Let the confidence interval be $[a,b]=\left[\hat{p}-z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p}+z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right]$

Then $\hat{p}=\frac{a+b}{2}$. For (0.408, 0.558), $\hat{p}=\frac{0.408+0.558}{2}=\mathbf{0.483}$

b

$$2\times z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}=b-a$$

$$\text{Margin of error }=z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}=\frac{b-a}{2}$$

$$=\frac{0.558-0.408}{2}$$

$$=\mathbf{0.075}$$

c

$$\text{Standard error } z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}=\frac{b-a}{2}$$

$$\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}=\frac{0.075}{1.645}=\mathbf{0.046}$$

d

$$\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}=0.046$$

$$\sqrt{\frac{0.483(1-0.483)}{n}}=0.046$$

Solve for n.

$n\approx 118$

Question 8

Let the confidence interval be $[a, b] = \left[\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right]$

$$\hat{p} = \frac{0.148 + 0.652}{2} = 0.4$$

$$E = \frac{0.652 - 0.148}{2} = 0.252$$

$$\text{SD}(\hat{p}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.09798$$

$$E = z \times SD(\hat{p})$$

Thus $z = \frac{E}{SD(\hat{p})} = \frac{0.252}{0.09798} = 2.5720$

Hence we have $\text{P}(-2.572 < z < 2.572)$

Using CAS, this corresponds to a probability of 0.98993…, or approximately **99%**.

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Question 9

Find E.

Let the confidence interval be $E = \frac{0.10}{2} = 0.05$

$\hat{p} = 28\% = 0.28$

For 95% confidence level, $z = 1.960$

Use $E = z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ to find n.

$$1.960\sqrt{\frac{0.28(1-0.28)}{n}} = 0.05$$

Solve for n to get $n = 309.7866\ldots$

Take $\boldsymbol{n \geq 310}$

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Question 10

a Assume the worst-case scenario. $\hat{p} = 0.5$ gives the largest SD.

Write the margin of error.

$E \approx 30\% = 0.03$

Write the formula.

$$E = z\sqrt{\frac{\hat{p}\left(1-\hat{p}\right)}{n}}$$

Substitute values and solve for n.

$$0.03 = 1.960\sqrt{\frac{0.5\left(1-0.5\right)}{n}}$$

Solve for n.

$$n = \frac{1.960^2 \times 0.25}{0.03^2} \approx 1067.111$$

At least **1068** consumers should be surveyed.

b Assume the worst-case scenario. $\hat{p} = 0.5$ gives the largest SD.

Write the margin of error.

$E \approx 10\% = 0.1$

Write the formula.

$$E = z\sqrt{\frac{\hat{p}\left(1-\hat{p}\right)}{n}}$$

Substitute values and solve for n.

$$0.1 = 1.645\sqrt{\frac{0.5\left(1-0.5\right)}{n}}$$

Solve for n.

$n \approx 67.65$

At least **68** consumers should be surveyed.

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Question 11

a Find the sample proportion.

$$\hat{p} \approx \frac{36}{75} = 0.48$$

State the value of $z_{0.95} \approx 1.960$.

$$\text{Confidence interval} \approx \left[\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right]$$

$$= \left[0.48 - 1.96\sqrt{\frac{0.48(1-0.48)}{75}}, 0.48 + 1.96\sqrt{\frac{0.48(1-0.48)}{75}}\right]$$

$$\approx [0.3669, 0.5931]$$

Since 0.4 ∈ [0.3669, 0.5931], there is insufficient evidence to suggest that the claimed value of *p* shouldn't be accepted. It could be 40%, but it cannot be known for certain.

b Find the sample proportion.

$$\hat{p} \approx \frac{50}{120} = \frac{5}{12}$$

State the value of $z_{0.95} \approx 1.960$.

$$\text{Confidence interval} \approx \left[\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right]$$

$$= \left[\frac{5}{12} - 1.960\sqrt{\frac{\frac{5}{12}\left(1-\frac{5}{12}\right)}{120}}, \frac{5}{12} + 1.960\sqrt{\frac{\frac{5}{12}\left(1-\frac{5}{12}\right)}{120}}\right]$$

$$\approx [0.328, 0.505]$$

The 95% confidence interval for the piano requiring new strings is about [0.328, 0.505].

Given that $0.30 \in [0.328, 0.505]$ there is insufficient evidence to suggest that the piano tuning demands north and south of the Swan River are the same.

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Question 12

a Find the sample proportion.

$\hat{p} = 40\% = 0.4$

State the value of $z_{0.95} \approx 1.960$.

$$\text{Confidence interval} \approx \left[\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right], \text{ where } n = 30$$

$$= \left[0.4 - 1.960\sqrt{\frac{0.4(1-0.4)}{30}}, 0.4 + 1.960\sqrt{\frac{0.4(1-0.4)}{30}}\right]$$

$$= [\mathbf{0.2247, 0.5753}]$$

b $\hat{p} = \frac{12}{50} = 0.24$

$$\text{SD}(\hat{p}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.4(1-0.4)}{30}} = 0.08944$$

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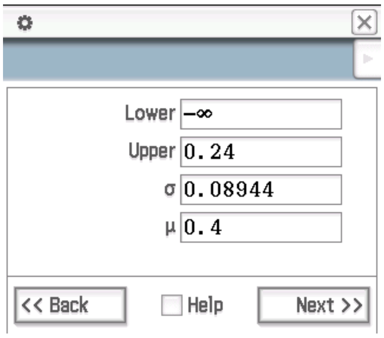

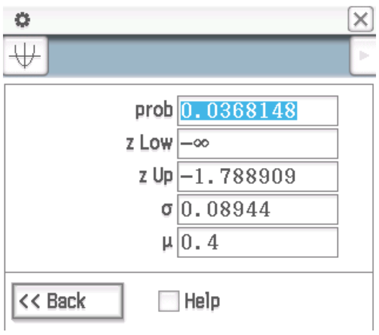

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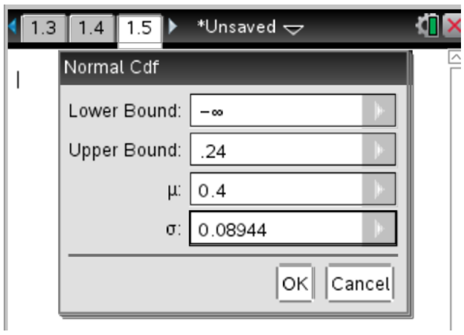

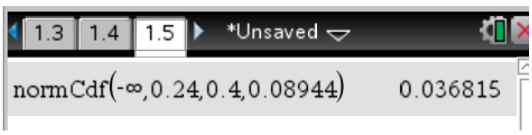

$P(\hat{p} < 0.24) = 0.0368$

There is a 3.68% chance of randomly selecting a sample where less than 24% of the people surveyed were in favour of adopting a new state flag.

c **A 95% confidence interval using $\hat{p} = 0.24$ and $n = 50$ gives [0.1216, 0.3584].**

Given that the two confidence intervals overlap with 0.24 ∈ [0.2247, 0.5753], there is insufficient evidence to suggest that the claim is true and that the newly proposed designs weren't better than the current.

Question 13

The formula for finding the approximate confidence interval is

$$\left[\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right].$$

Based on Albin's 18 tosses, the sample probability of tossing a head is $\boldsymbol{\hat{p} = \frac{12}{18} = \frac{2}{3}}$.✓

The number of tosses is $n = 18$ and the z value is given as $z = 1.645$.

Hence we have

$$\left[\frac{2}{3} - 1.645\sqrt{\frac{\frac{2}{3}\left(1-\frac{2}{3}\right)}{18}}, \frac{2}{3} + 1.645\sqrt{\frac{\frac{2}{3}\left(1-\frac{2}{3}\right)}{18}}\right]$$

$$= \left[\frac{2}{3} - 1.645\sqrt{\frac{\frac{2}{3}\left(\frac{1}{3}\right)}{18}}, \frac{2}{3} + 1.645\sqrt{\frac{\frac{2}{3}\left(\frac{1}{3}\right)}{18}}\right]$$

$$= [\mathbf{0.4839, 0.8494}]$$

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Question 14 [SCSA MM2017 Q4] (3 marks)

(✓= 1 mark)

$$z\sqrt{\frac{\hat{p}(1-\hat{p})}{n_2}}=\frac{1}{2}z\sqrt{\frac{\hat{p}(1-\hat{p})}{n_1}}$$

$$\frac{\hat{p}(1-\hat{p})}{n_2}=\frac{1}{4}\frac{\hat{p}(1-\hat{p})}{n_1}$$

$$\frac{n_2}{n_1}=4$$

uses formula for margin of error to relate the two sample sizes✓

simplifies equation by squaring both sides and cancelling✓

rearranges to find ratio✓

Question 15 [SCSA MM2018 Q5] (3 marks)

(✓= 1 mark)

The width of a confidence interval is inversely proportional to the square root of sample size. Therefore, to have one third the width of the confidence interval requires a sample size nine times as large, so a sample size of 1800 is needed.

or

$w = \dfrac{z\sigma}{\sqrt{n}}$ (1)

sample size: $n_1 : \dfrac{w}{3} = \dfrac{z\sigma}{3\sqrt{n_1}}$ (2)

dividing (1): $\dfrac{w}{3} = \dfrac{z\sigma}{3\sqrt{n}}$

substituting for $\dfrac{w}{3}$ into (2): $\dfrac{z\sigma}{3\sqrt{n}} = \dfrac{z\sigma}{\sqrt{n_1}}$

solving for $n_1 : n_1 = 9n = 1800$.

uses the new width as $\mathbf{\dfrac{w}{3}}$ ✓

states that the sample size is nine times as large✓

gives correct value of sample size✓

or

obtains equation (1) and (2)✓

solves equation✓

obtains correct sample size✓

Question 16 (4 marks)

(✓= 1 mark)

a $\hat{p} = \dfrac{6}{100} = 0.06$ ✓

For a 95% confidence interval, $z = 1.960$

$$\left[\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right] \checkmark$$

$$\left[0.06 - 1.960\sqrt{\frac{0.06(1-0.06)}{100}}, 0.06 + 1.960\sqrt{\frac{0.06(1-0.06)}{100}}\right]$$

$$= [0.013, 0.107] \checkmark$$

b **The distribution of $\hat{p}$ may not be approximately normal, as $\hat{p} = 0.06$ and $np = 6 < 10$, meaning that the distribution may be positively skewed and, hence, a normal distribution may not be appropriate.** ✓

or

$0.06 - 3\text{SD}(\hat{p}) = 0.06 - 3(0.0237) = 0$ and so, a normal distribution is not appropriate.

Question 17 (3 marks)

(✓ = 1 mark)

The formula for finding the approximate confidence interval is

$$\left[\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right] \checkmark$$

The standard normal quantile associated with a 90% confidence interval is 1.645.

The value of $\hat{p}$ is 0.76 and $n = 574$.

Substitute these values in the formula.

$$\left[0.76 - 1.645\sqrt{\frac{0.76 \times 0.24}{574}}, 0.76 + 1.645\sqrt{\frac{0.76 \times 0.24}{574}}\right] \checkmark$$

$$= [0.731, 0.789] \checkmark$$

Question 18 [SCSA MM2016 Q14c] (3 marks)

(✓= 1 mark)

Smallest margin of error occurs for smallest confidence percentage 90%.

There is a trade-off between level of confidence and margin of error.

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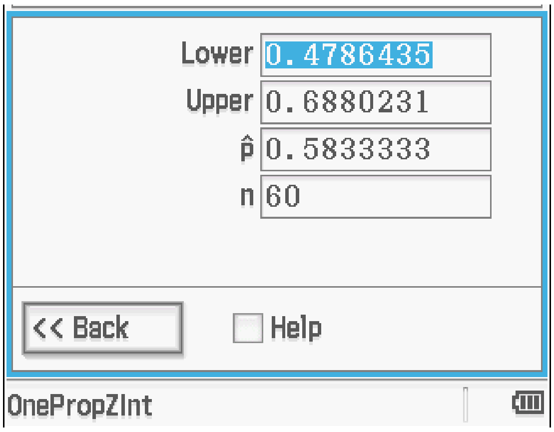

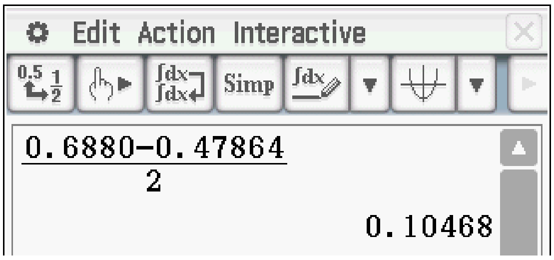

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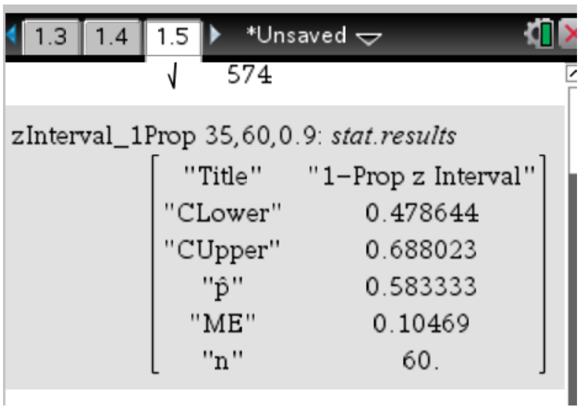


uses 90% confidence✓

states trade-off between confidence and margin of error✓

determines margin of error✓

Question 19 [SCSA MM2017 Q18abc] (6 marks)

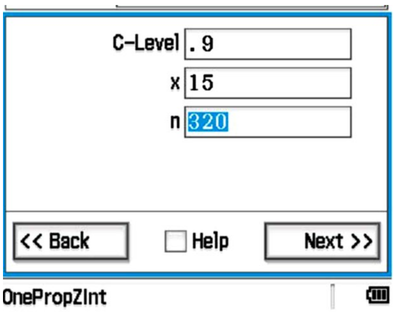
(✓= 1 mark)

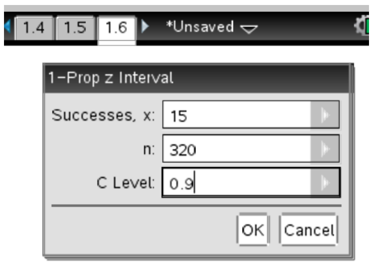
a $\frac{15}{320} = 0.046875 = 4.69\%$

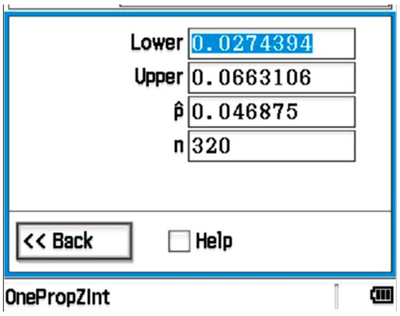
calculates proportion✓

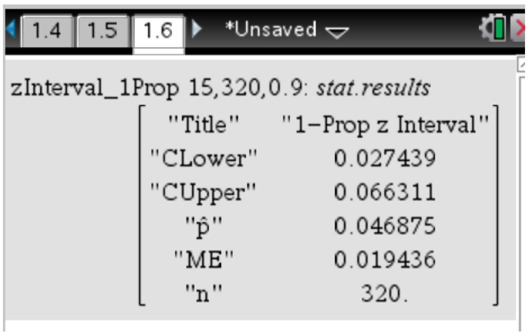
b $$\left[\frac{15}{320} - 1.645\sqrt{\frac{\left(\frac{15}{320}\right)\left(1-\frac{15}{320}\right)}{320}}, \frac{15}{320} + 1.645\sqrt{\frac{\left(\frac{15}{320}\right)\left(1-\frac{15}{320}\right)}{320}}\right] \approx [0.0274, 0.0663]$$

ClassPad **TI-Nspire**









Confidence interval: **[0.0274, 0.0663]**

uses $z \approx 1.645$✓

calculates confidence interval✓

rounds to four decimal places✓

c $1.645\sqrt{\frac{\left(\frac{15}{320}\right)\left(1-\frac{15}{320}\right)}{320}} \approx 0.0194$

substitutes into formula✓

calculates standard error✓

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Question 20 [SCSA MM2018 Q13] (10 marks)

(✓= 1 mark)

a $E=\sqrt{\hat{p}(1-\hat{p})}$

$\frac{dE}{d\hat{p}}=\frac{1-2\hat{p}}{2\sqrt{\hat{p}(1-\hat{p})}}$

$0=1-2\hat{p}$

$\hat{p}=0.5$

$\left.\frac{d^2E}{d\hat{p}^2}\right|_{\hat{p}=0.5}=-2\Rightarrow \text{maximum}$

differentiates E with respect to $\hat{p}$ ✓

equates derivative to zero and solves for $\hat{p}$ ✓

uses second derivative r sign test to confirm maximum✓

b Use $\hat{p}=0.5$

z value for 99% = 2.576

E for sample proportion $E=z\sqrt{\frac{p(1-p)}{n}}$

and $E=0.08$

$0.08=2.576\sqrt{\frac{0.5(1-0.5)}{n}}$

$n=259.21$

Hence 260 vehicles should be tested.

uses $\hat{p}=0.5$ and z-value✓

equates standard error to 0.08✓

solves for n and rounds up to 260✓

c Use

$$\hat{p} = \frac{0.342 + 0.558}{2} = 0.45$$

$$E = \frac{0.558 - 0.45}{2} = 0.108$$

$$E = z\sqrt{\frac{p(1-p)}{n}}$$

$$0.108 = 2.576\sqrt{\frac{0.45(1-0.45)}{n}}$$

$$n = 141$$

Number of vehicles with incorrect towing capacity = np

$= 141 \times 0.45$

≈ 63

finds correct $\hat{p}$ ✓

finds correct E✓

finds number in sample✓

finds number of vehicles with incorrect towing capacity✓

Question 21 [SCSA MM2019 Q14] (7 marks)

(✓= 1 mark)

a $n = \dfrac{1.960^2 \times 0.5 \times 0.5}{0.01^2} = 9604$

uses correct *z*-critical value✓

uses correct formula✓

gives correct value to nearest integer rounded up✓

b Any two of:

1. Sample size: as sample size increases the width decreases.
2. Sample proportion: as sample proportion moves away from 0.5 the width decreases.
3. Confidence level: as confidence level increases the width increase.

states one correct reason✓

states the effect✓

states two correct reasons✓

states effect✓

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Question 22 [SCSA MM2020 Q14ab MODIFIED] (6 marks)

(✓= 1 mark)

a

$$n > \left(\frac{1.960\sqrt{0.5\times 0.5}}{0.03}\right)^2 = 1067.111$$

So the minimal sample size is 1068.

uses the correct *z* value✓

uses 0.5 in the expression for standard error✓

determines the sample size (as an integer)✓

b

$$\varepsilon = 2.576\times\sqrt{\frac{0.5\times 0.5}{500}} = 0.058$$

That is, within 5.8%

uses the correct *z* value✓

uses 0.5 in the expression for standard error✓

calculates the error✓

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Question 23 [SCSA MM2018 Q17efg] (6 marks)

(✓= 1 mark)

a

$$95\%\text{ confidence interval} = \left[0.5 - 1.960 \times \sqrt{\frac{0.5 \times 0.5}{500}}, 0.5 + 1.960 \times \sqrt{\frac{0.5 \times 0.5}{500}}\right]$$

$$= [0.5 - 0.04383, 0.5 + 0.04383]$$

$$= [0.4562, 0.5438]$$

uses the correct value for the standard error✓

uses the correct z-value interval✓

calculates the confidence interval to 4 decimal places✓

b Since 0.6 is not contained in the 95% confidence interval, it is unlikely that Tina is correct.

refers to 0.6 not being in the interval✓

concluding that it is unlikely that Tina is correct✓

c Tina should take another random and obtain another 95% confidence interval.

states answer✓

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Question 24 (7 marks)

(✓ = 1 mark)

a The sample proportion $\hat{p} = \frac{10}{100} = 0.1$ ✓

$z_{0.99} = 2.576$ ✓, $n = 100$.

Confidence interval $\approx \left[\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right]$

$$= \left[0.1 - 2.576\sqrt{\frac{0.1(1-0.1)}{100}}, 0.1 + 2.576\sqrt{\frac{0.1(1-0.1)}{100}}\right]$$

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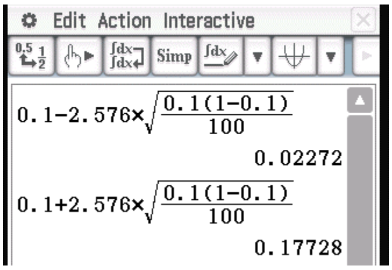

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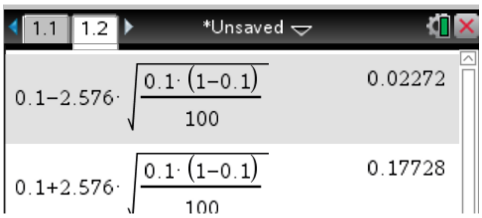

Rounding to four decimal places the $= [0.0227, 0.1773]$ ✓

b The sample proportion $\hat{p} = \frac{10}{100} = 0.1$ ✓

$z_{0.90} = 1.645$ ✓, $n = 100$.

$$\text{Confidence interval} \approx \left[\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right]$$

$$= \left[0.1 - 1.645\sqrt{\frac{0.1(1-0.1)}{100}}, 0.1 + 1.645\sqrt{\frac{0.1(1-0.1)}{100}}\right]$$

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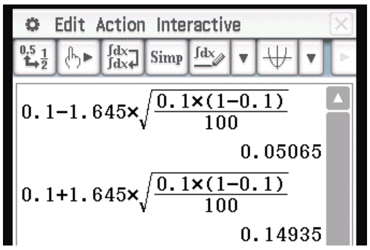

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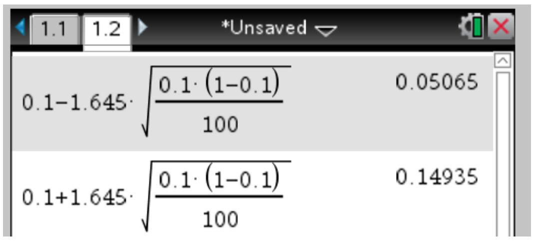

Rounding to four decimal places the $= [\mathbf{0.0507, 0.1494}]$ ✓

c Given that 6% (0.06) lies in both confidence intervals, then there is insufficient evidence to suggest that the company's historical data is no longer relevant.

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Question 25 (4 marks)

(✓= 1 mark)

a The sample proportion $\hat{p} = \frac{40}{500} = 0.08$ ✓

$z_{0.95} = 1.960$ ✓

$$\text{Confidence interval} \approx \left[\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right]$$

$$= \left[0.08 - 1.960\sqrt{\frac{0.08(1-0.08)}{500}}, 0.08 + 1.960\sqrt{\frac{0.08(1-0.08)}{500}}\right]$$

$$= [0.0562, 0.1038]$$ ✓

b **Given that 0.05 does not lie in the confidence interval, then there is insufficient evidence to suggest that the company's historical data is still relevant and it may be outdated, but it cannot be known for certain.**✓

Question 26 (7 marks)

(✓ = 1 mark)

a $\hat{p} = \dfrac{43}{1000} = \mathbf{0.043}$, $\boldsymbol{n} = \mathbf{1000}$✓

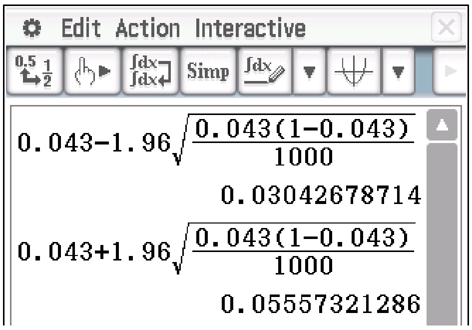
$$\text{SD}(\hat{p}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.043(1-0.043)}{1000}} = 0.00641 \checkmark$$

Given that n is sufficiently large and $np = 43 \geq 10$ and $n(1 - p) = 957 \geq 10$, then $\hat{p}$ is approximately normal such that $\hat{p} \sim N(\mathbf{0.043}, \mathbf{0.00641^2})$.✓

b For a 95% confidence interval, $z = 1.960$.

Use the formula $\hat{p} \mp z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$ **to calculate the confidence interval.**✓

ClassPad



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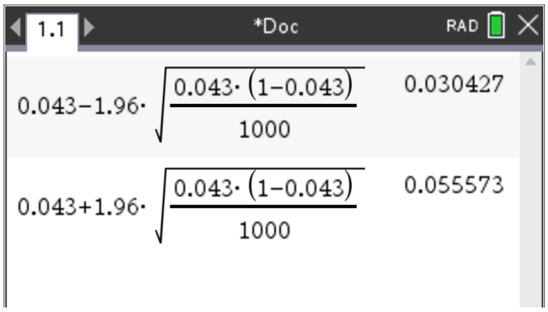

Rounding to three decimal places, the confidence interval is $[\mathbf{0.030}, \mathbf{0.056}]$.✓

c **Even though 0.04 lies within the confidence interval [0.030, 0.056]✓, it cannot be inferred that the reviewer visited that specific hall.**✓

Question 27 [SCSA MM2019 Q8] (7 marks)

(✓= 1 mark)

a $\hat{p} = \dfrac{56}{250} = 0.224$

calculates correct proportion ✓

b $$\left[0.224 - 1.960\sqrt{\frac{0.224(1-0.224)}{250}},\ 0.224 + 1.960\sqrt{\frac{0.224(1-0.224)}{250}}\right] = [0.1723, 0.2757]$$

uses z = 1.960✓

calculates confidence interval✓

rounds to four decimal places✓

c $$E = 1.960\sqrt{\frac{0.224(1-0.224)}{250}}$$
$$= 0.0517$$

or

$$E = \frac{0.2757 - 0.1723}{2}$$
$$= 0.0517$$

calculates margin of error✓

d No. Only 95% of confidence intervals are expected to contain the true proportion. It is possible that the survey and calculation by the junior staff member was performed appropriately, but happened to yield one of the 5% of confidence intervals that do not contain the true proportion.

answers ‘No’ with a reference to part a✓

justifies answer by saying that only 95% of intervals are expected to contain the true proportion✓

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Question 28 [SCSA MM2021 Q13defg] (8 marks)

(✓= 1 mark)

a (0.04, 0.16) is the 95% confidence interval as it is the wider of the two intervals provided (the 95% confidence interval is wider than the 90% confidence interval).

chooses the correct interval✓

provides correct justification for the choice✓

b The mid-point of the confidence intervals gives $\hat{p} = 0.1$. Since 100 games were observed it means that 0.1 × 100 = 10 wins were observed.

determines the value of $\hat{p}$ ✓

determines the number of wins observed✓

c The width of the confidence interval is proportional to $\sqrt{\frac{1}{n}}$.

Hence increasing the number of observed games by a factor of 4 will lead to the confidence interval width reducing by a factor of 2 (i.e. halved).

states that the width will reduce✓

determines that the reduction is by a factor of 2✓

d A mistake has not necessarily been made. Not all 90% or 95% confidence intervals will contain the true proportion p.

states that a mistake has not necessarily been made✓

states that not all confidence intervals contain the true population proportion✓

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Question 29 [SCSA MM2016 Q10 MODIFIED] (11 marks)

(✓= 1 mark)

a

$$\hat{p} = \frac{986}{1450} = 0.68$$

$$SD(\hat{p}) = \sqrt{\frac{0.68(1-0.68)}{1450}} = 0.01225$$

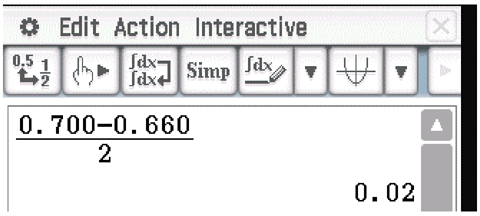
$$0.68 - 1.645(0.01225) \le \hat{p} \le 0.68 + 1.645(0.01225)$$
$$0.6598 \le \hat{p} \le 0.7001$$
$$0.660 \le \hat{p} \le 0.700$$

states that sample proportions form a normal distribution✓

determines confidence interval✓

expresses interval rounded to three decimal places✓

b **ClassPad**



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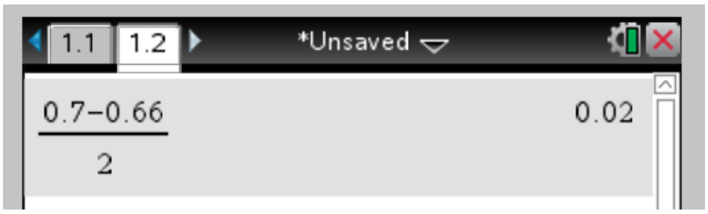


The margin of error is 0.02

uses confidence interval✓

determines margin of error✓

c Survey 2

$$\hat{p} = \frac{1772}{3221} \approx 0.5501$$

$$\text{confidence interval} = \left[0.5501 - 1.645\sqrt{\frac{0.5501(1-0.5501)}{3221}}, 0.5501 + 1.645\sqrt{\frac{0.5502(1-0.5501)}{3221}}\right]$$

$$= [0.536, 0.565]$$

Survey 3

$$\hat{p} = \frac{1021}{1566} \approx 0.6520$$

$$\text{confidence interval} = \left[0.6520 - 1.645\sqrt{\frac{0.6520(1-0.6520)}{1566}}, 0.6520 + 1.645\sqrt{\frac{0.6520(1-0.6520)}{1566}}\right]$$

$$= [0.632, 0.672]$$

Survey 4

$$\hat{p} = \frac{2203}{3221} \approx 0.6839$$

$$\text{confidence interval} = \left[0.6839 - 1.645\sqrt{\frac{0.6839(1-0.6839)}{3221}}, 0.6839 + 1.645\sqrt{\frac{0.6839(1-0.6839)}{3221}}\right]$$

$$= [0.670, 0.697]$$

calculates one correct confidence interval✓

calculates two correct confidence intervals✓

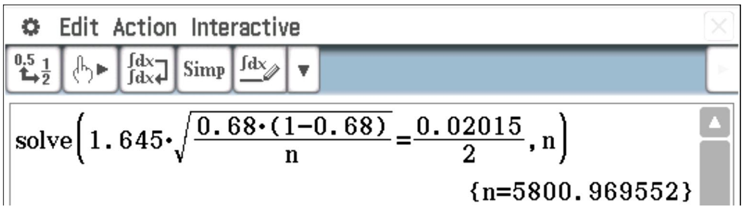
calculates three correct confidence intervals✓

d Given that there is no overlap between the confidence intervals for Survey 1 and Survey 2, there is sufficient evidence to suggest that the samples may have come from different populations, i.e., that Survey 2 did not come from Western Australia, but we cannot say for certain.

states that there is no overlap between the confidence intervals for Survey 1 and Survey 2 ✓

states that the claim is not conclusive✓

e **ClassPad**



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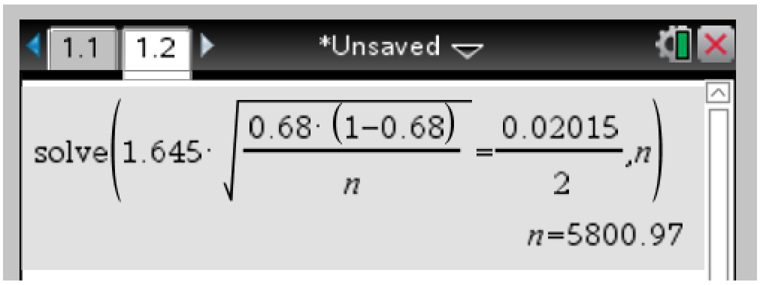


Rounds up to $n = 5800$

sets up an equation to solve for sample size✓

rounds correct sample size to nearest integer✓

Question 30 [SCSA MM2016 Q20 MODIFIED] (12 marks)

(✓= 1 mark)

a $X \sim \text{Bin}(30, 0.8)$

$\mu = np = 24$

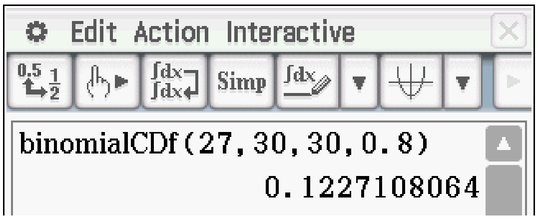
$s = \sqrt{\dfrac{0.8(1-0.8)}{30}} = 0.073$

identifies binomial distribution✓

determines mean✓

determines standard deviation✓

b **ClassPad**



TI-Nspire

So the probability is 0.1227

determines probability✓

rounds to four decimal places✓

c Given that n is sufficiently large and $np = 450 \geq 10$ and $n(1-p) = 150 \geq 10$, then $\hat{p}$ will be approximately normally distributed.

Then $\hat{p} = \dfrac{450}{600} = 0.75$

95% confidence interval

$$\left[0.75 - 1.960\sqrt{\frac{0.75(1-0.75)}{600}}, 0.75 + 1.960\sqrt{\frac{0.75(1-0.75)}{600}}\right]$$

$$= [0.7154, 0.7846]$$

states $np \geq 10$ and $n(1-p) \geq 10$✓

determines sample proportion✓

uses correct expression for confidence interval✓

finds correct confidence interval✓

d Given that $0.8 \notin [0.7154, 0.7846]$, then there is insufficient evidence to suggest that there isn't an error with the machine's calibration and it may be off, but it cannot be known for certain.

explains that it cannot be conclusively stated the machine is faulty because the sample proportion lies outside the confidence interval✓

e Not all confidence intervals are expected to contain the true value of p and it is expected that 95% of confidence intervals to be constructed will contain p. It can be due to the nature of random sampling that 0.8 did not lie in the first confidence interval.

state that $\hat{p}$ may not lie in the confidence interval✓

suggest that this can occur because of random sampling✓

Cumulative examination: Calculator-free

Question 1 (4 marks)

(✓ = 1 mark)

a The mean is np and the variance is $np(1-p)$.

Hence $\mathbf{np=20}$ ✓ and $\mathbf{np(1-p)=4}$ ✓

b $4=20(1-p)$

$1-p=\frac{1}{5}$

$\mathbf{p=\frac{4}{5}}$ ✓

$n\times\frac{4}{5}=20$

$\mathbf{n=25}$ ✓

Question 2 (5 marks)

(✓ = 1 mark)

a $\frac{dy}{dx}=x^3\times\left(\frac{d}{dx}(\ln(3x))\right)+\frac{d}{dx}(x^3)\times\ln(3x)$ ✓ product rule

$=x^3\times\frac{1}{x}+3x^2\times\ln(3x)$

$\mathbf{=x^2+3x^2\ln(3x)}$ ✓

b $\frac{d}{dx}(x^3\ln(3x))=x^2+3x^2\ln(3x)$ ✓ from part **a**

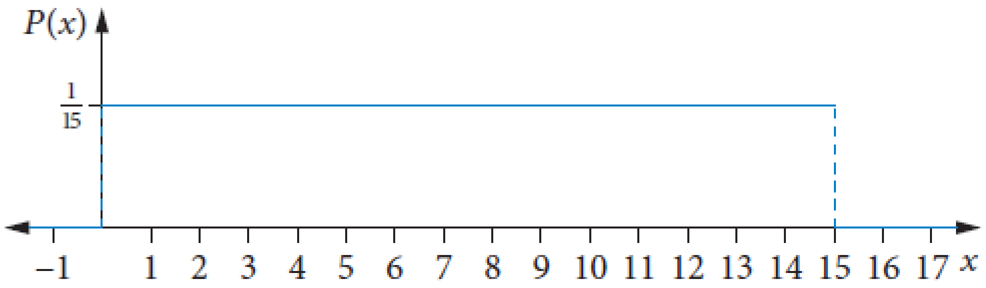
$x^3\ln(3x)=\int x^2dx+3\int x^2\ln(3x)\,dx$ ✓

$\int x^2\ln(3x)\,dx=\frac{1}{3}x^3\left(\ln(3x)-\frac{1}{3}\right)+c$ or $\frac{1}{3}x^3\ln(3x)-\frac{1}{9}x^3+c$ ✓

Question 3 [SCSA MM2017 Q2] (6 marks)

(✓ = 1 mark)

a



draws uniform distribution correctly✓

includes $P(x)$ axes scale✓

b $\frac{3}{15}$

states correct probability✓

c $\frac{7}{15}$

states correct probability✓

d $$\frac{\frac{5}{15}}{\frac{8}{15}} = \frac{5}{8}$$

correctly determines numerator✓

correctly determines denominator✓

Question 4 [SCSA MM2020 Q7 MODIFIED] (13 marks)

(✓ = 1 mark)

a Solve $f(x) = 0$

$0 = (e^x - 2)(e^x - 4)$

$e^x = 2, e^x = 4$

$x = \ln(2), x = \ln(4)$

Hence x-intercepts are at (ln(2), 0) and (ln(4), 0)

states correct equation to be solved✓

solves for *x*✓

states coordinates of both points✓

b $f'(x) = 2e^{2x} - 6e^x$

Solve $f'(x) = 0$

$0 = 2e^{2x} - 6e^x$

$0 = 2e^x(e^x - 3)$

$e^x = 3$

$x = \ln(3)$

Substitute $x = \ln(3)$ into $f(x)$

$$\begin{aligned} f(\ln(3)) &= e^{2\ln(3)} - 6e^{\ln(3)} + 8 \\ &= e^{\ln(9)} - 6e^{\ln(3)} + 8 \\ &= 9 - 6 \times 3 + 8 \\ &= -1 \end{aligned}$$

Turning point at (ln(3), −1)

differentiates *f* (*x*) correctly and equates to 0✓

shows the steps required to solve for *x*✓

demonstrates the use of log laws to determine the *y*-coordinate✓

c Solve $f''(x)=0$

$$f''(x)=4e^{2x}-6e^{x}$$
$$0=4e^{2x}-6e^{x}$$
$$0=2e^{x}\left(2e^{x}-3\right)$$
$$2e^{x}=3$$
$$e^{x}=\frac{3}{2}$$
$$x=\ln\left(\frac{3}{2}\right)$$

Substitute $x=\ln\left(\frac{3}{2}\right)$ into $f(x)$

$$f\left(\ln\left(\frac{3}{2}\right)\right)=e^{2\left(\ln\left(\frac{3}{2}\right)\right)}-6e^{\ln\left(\frac{3}{2}\right)}+8$$
$$=\left(\frac{3}{2}\right)^{2}-6\times\frac{3}{2}+8$$
$$=\frac{9}{4}-9+8$$
$$=\frac{5}{4}$$

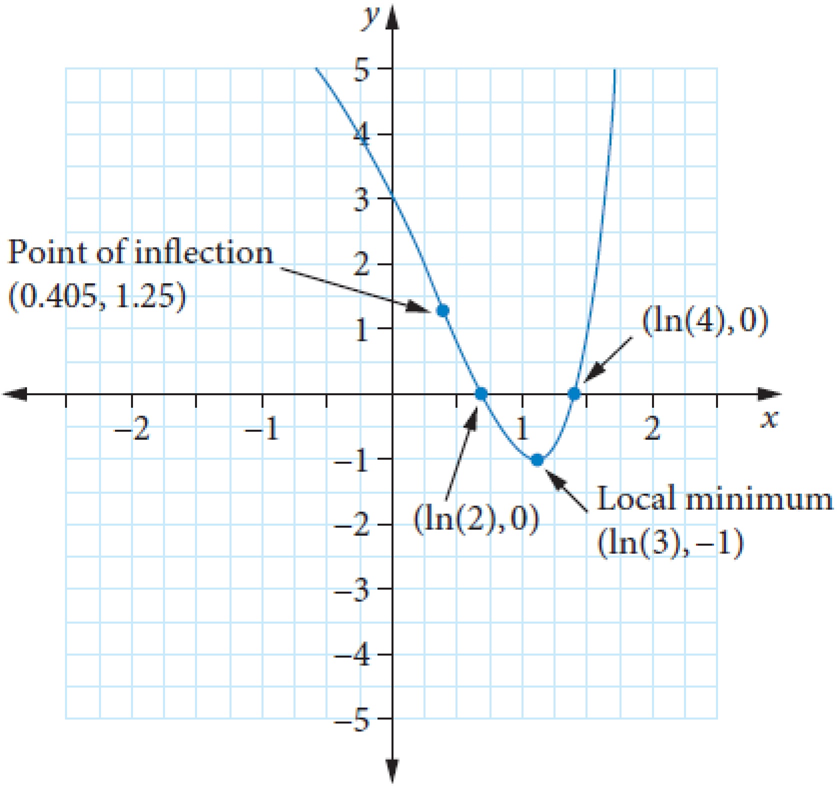
Inflection point at $\left(\ln\left(\frac{3}{2}\right),\frac{5}{4}\right)$

differentiates $f(x)$ correctly and equates to 0✓

solves for x✓

determines y-coordinate of inflection point✓

d



intercepts correct and labelled✓

turning point and inflection point correct and labelled✓

concavity correct✓

limiting behaviour correct✓

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Cumulative examination: Calculator-assumed

Question 1

(✓ = 1 mark)

$$V = \frac{1}{6}\pi x^3$$

$$\mathbf{200 = \frac{1}{6}\pi x^3 \Rightarrow x = \sqrt[3]{\frac{1200}{\pi}} = 7.25566} ✓$$

$$\mathbf{\frac{dV}{dx} \approx \frac{\delta V}{\delta x} \Rightarrow \delta x = \frac{dx}{dV}\delta V} ✓$$

$\frac{dV}{dx} = \frac{\pi}{2}x^2$, so $\delta x = \frac{2}{\pi x^2}\delta V$, with $\delta V = 210 - 200 = 10$

$$\delta x \approx \frac{2}{\pi(7.25566)^2} \times 10 = 0.1209$$

Change in depth is approximately 0.12 cm.✓

Question 2 (7 marks)

(✓ = 1 mark)

a $f(x)=\frac{1}{5}(x-2)^2(5-x)=\frac{1}{5}(-x^3+9x^2-24x+20)$

$f'(x)=\frac{1}{5}(-3x^2+18x-24)=-\frac{3}{5}(x^2-6x+8)$

$\boldsymbol{f'(x)=\frac{1}{5}(-3x^2+18x-24)=-\frac{3}{5}(x^2-6x+8)=-\frac{3}{5}(x-4)(x-2)}$ ✓, or use product rule.

b **i** Gradient at $x = 1$ is $m=f'(1)=-\frac{3}{5}(1-6+8)=-1.8$

Equation of tangent is $y=mx+c=-1.8x+c$

Use $\left(1,\frac{4}{5}\right)$ to find c.

$0.8=-1.8(1)+c \Rightarrow c=2.6$

Hence $\boldsymbol{y=-1.8x+2.6}$ ✓

ii Q is the x-intercept of the gradient function, so solve $-1.8x+2.6=0$.

This gives $x=\frac{2.6}{1.8}=\frac{13}{9}$.

The coordinates of Q are $\left(\frac{13}{9},0\right)$ ✓

S is the y-intercept of the gradient function, so when $x = 0$, solve $y=-1.8(0)+2.6=2.6$.

The coordinates of S are $\left(0,\frac{13}{5}\right)$ ✓

c First, find the x-value of the point of intersection of $f(x)$ and the straight line $y=-1.8x+2.6$. (The first value is $x = 1$).

Solve $\frac{1}{5}(x-2)^2(5-x)=-1.8x+2.6$ to get $\boldsymbol{x=1, x=7}$ ✓

Hence calculate $\int_1^7\left(\frac{1}{5}(x-2)^2(5-x)-(-1.8x+2.6)\right)dx$ ✓ to get 21.6

The shaded area is 21.6 units^2✓

Question 3 (9 marks)

(✓ = 1 mark)

a **i** The chance of correctly answering a question randomly is $\frac{1}{4}$ or 0.25.

Let X represent the number of questions Steve answers correctly.

Then we have $X \sim \text{Bin}(3, 0.25)$, which is $\frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} = \mathbf{\frac{1}{64}}$ ✓

ii $X \sim \text{Bin}(20, 0.25)$ ✓

ClassPad

binomialCDf(10, 20, 20, 0.25)
0.01386441694

TI-Nspire

The probability is 0.0139✓

iii $\text{Var}(X) = np(1-p)$

$$\mathbf{\frac{75}{16} = n \times \frac{1}{4} \times \frac{3}{4}} \checkmark$$

$$\mathbf{n = \frac{75}{3} = 25}$$

b The first question is answered incorrectly, given.

Favourable outcomes after that (getting questions 3, 4 and 5 correct) are: rrrr and wrrr, where w = wrong answer, r = right answer

$\left(\frac{1}{3}\right)\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\left(\frac{3}{4}\right) = \mathbf{\frac{9}{64}}$ ✓ and $\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\left(\frac{3}{4}\right)\left(\frac{3}{4}\right) = \mathbf{\frac{1}{8}}$ ✓

w r r r r   w w r r r

$$\frac{9}{64} + \frac{1}{8} = \frac{9+8}{64} = \mathbf{\frac{17}{64}} \checkmark$$

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c $X \sim \text{Bin}(25, p)$

$P(Y > 23) = 6P(Y = 25)$, given

Hence $\mathbf{P(Y = 24) + P(Y = 25) = 6P(Y = 25)}$✓

$P(Y = 24) - 5P(Y = 25) = 0$

$$^{25}C_{24}p^{24}(1-p)^1 - 5\times {}^{25}C_{25}p^{25}(1-p)^0 = 0$$
$$25p^{24}(1-p) - 5p^{25} = 0$$
$$25p^{24} - 25p^{25} - 5p^{25} = 0$$
$$25p^{24} - 30p^{25} = 0 \quad ✓$$
$$5p^{24}(5-6p) = 0$$
$$p = 0,\ p = \frac{5}{6}$$

Since p > 0, take $p = \frac{5}{6}$

Question 4 (9 marks)

(✓ = 1 mark)

a Let $t = 0$, which represents before the start of the course.

$w = 100\ln(1) + 150 =$ **150 words**✓

b Let $t = 1$

$$\boldsymbol{w = 100\ln(1+1) + 150 = 100\ln(2) + 150 = 219.314...}$$ ✓

w is the number of words he learns, so he has learnt approximately **219 words**✓ on the first day.

c Let $t = 5$

$$\boldsymbol{w = 100\ln(5+1) + 150 = 100\ln(6) + 150 = 329.175...}$$ ✓

Approximately 329 words were learnt after 5 days. When these are added to the 150 he originally knew, he now knows 329 + 150 = **479**✓ **words** known after 5 days.

d Require the value of t such that $\boldsymbol{w \geq 600 - 150 \ (= 450)}$.✓.

450 words must be learnt.

$100\ln(t+1) + 150 \geq 450$

$\ln(t+1) \geq 3$

$t + 1 \geq e^3$

$t \geq e^3 - 1$

$t \geq 19.085...$

It will take 20 days.✓

e $w = 100\ln(t+1) + 150$

$$\boldsymbol{\ln(t+1) = \frac{w - 150}{100}}$$ ✓

$$\boldsymbol{t + 1 = e^{\frac{w-150}{100}}}$$ ✓

$$\boldsymbol{t = e^{\frac{w-150}{100}} - 1 = e^{0.01w - 1.5} - 1}$$ ✓

Question 5 (14 marks)

(✓ = 1 mark)

a **i** $y=\frac{1}{200}\left(ax^3+bx^2+c\right)$

Use (2, 0) to get $\mathbf{\frac{1}{200}\left(8a+4b+c\right)=0}$ ✓

$\frac{dy}{dx}=\frac{1}{200}\left(3ax^2+2bx\right)$

$\frac{dy}{dx}=0$ at $x=4$. Hence $\mathbf{\frac{dy}{dx}=\frac{1}{200}\left(48a+8b\right)=0}$ ✓

Gradient at (2, 0) is −0.06

Hence $\mathbf{\frac{dy}{dx}=\frac{1}{200}\left(3a(2)^2+2b(2)\right)=-0.06\Rightarrow\frac{1}{200}\left(12a+4b\right)=-0.06}$ ✓

ii $\frac{1}{200}\left(8a+4b+c\right)=0$

$\frac{1}{200}\left(48a+8b\right)=0$ **use the appropriate function on the calculator**✓

$\frac{1}{200}\left(12a+4b\right)=-0.06$

Solve to obtain $a=1$, $b=-6$, $c=16$

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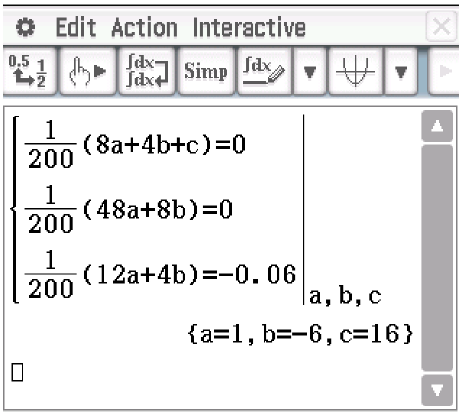
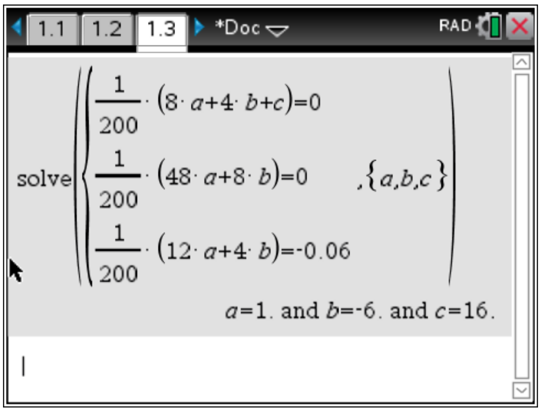


✓

TI-Nspire



✓

Or

Algebraically, the equations become

$\mathbf{8a+4b+c=0}$ **(1)**

$\mathbf{6a+b=0}$ **(2)** ✓

$\mathbf{3a+b=-3}$ **(3)**

(2) – (3) gives $3a = 3$, so $\boldsymbol{a = 1}$

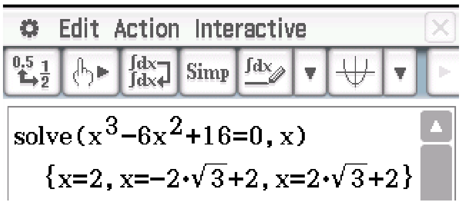
From (2), $6 + b = 0$, so $\boldsymbol{b = -6}$ ✓ show suitable working to obtain the correct values.

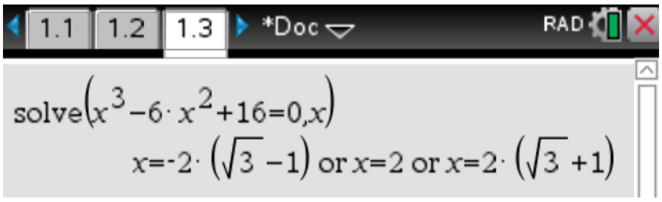
From (1), $8 - 24 + c = 0$, so $\boldsymbol{c = 16}$

b **i** M and P are the x-intercepts. To find them, let $y = 0$ and solve for x.

$$\boldsymbol{y=\frac{1}{200}\left(x^3-6x^2+16\right)=0} \checkmark$$

ClassPad **TI-Nspire**





The coordinates are $\boldsymbol{M\left(2\sqrt{3}+2,0\right)}$ and $\boldsymbol{P\left(-2\sqrt{3}+2,0\right)}$ ✓

ii Find the length MN.

Using $N(2,0), M\left(2\sqrt{3}+2,0\right)$, length MN is $2\sqrt{3}+2-2=2\sqrt{3}$

The length of the tunnel is $\boldsymbol{2\sqrt{3}}$ km✓

iii Find the y-coordinate of the minimum turning point.

From part **a**, the minimum occurs at $x = 4$.

$$y=\frac{1}{200}\left(4^3-6(4)^2+16\right)=-\frac{2}{25}.$$

The maximum depth is 0.08 km or 80 m✓

c $d = 0, v = w$

Hence $w=k\log_e\left(\frac{1}{7}\right)=-k\log_e(7)$

$$\boldsymbol{k=-\frac{1}{\log_e(7)}w=-\frac{w}{\log_e(7)}} \checkmark$$

d

$$\frac{120\log_e(2)}{\log_e(7)} = k\log_e\left(\frac{2.5+1}{7}\right)$$

$$\mathbf{\frac{120\log_e(2)}{\log_e(7)} = -\frac{w}{\log_e(7)}\log_e\left(\frac{1}{2}\right)} ✓$$

$$120\log_e(2) = w\log_e(2)$$

$\boldsymbol{w = 120}$✓

e

$v = 0$

$$0 = k\log_e\left(\frac{d+1}{7}\right)$$

$$\mathbf{\frac{d+1}{7} = 1} ✓$$

$d + 1 = 7$

$d = 6$

Distance from front of train to rock is **6.2 – 6 = 0.2 km**✓

Question 6 [SCSA MM2017 Q12bcde MODIFIED] (9 marks)

(✓ = 1 mark)

a $X \sim \text{Bin}(200, 0.01)$

identifies the binomial distribution✓

specifies correct parameters✓

b $P(X > 4) = 1 - P(X \le 4) = 1 - 0.9483 = 0.0517$

uses correct parameters for binomial✓

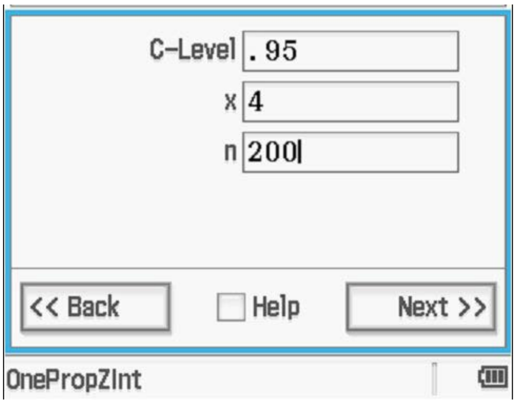
calculates correct probability✓

c $\hat{p} = \frac{4}{200} = 0.02$

$$95\% \text{ confidence interval} = \left[0.02 - 1.960 \times \sqrt{\frac{0.02 \times 0.98}{200}}, 0.02 + 1.960 \times \sqrt{\frac{0.02 \times 0.98}{200}}\right]$$

That is, $[0.0006, 0.0394]$

ClassPad



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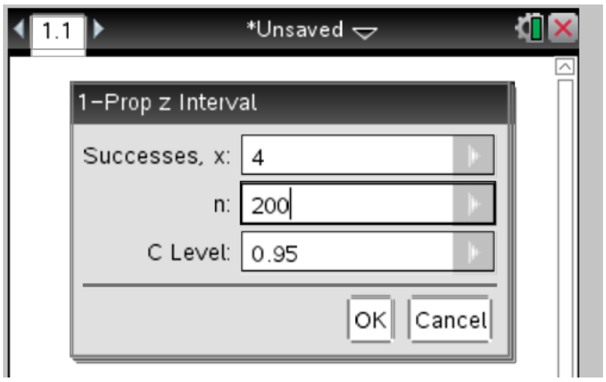

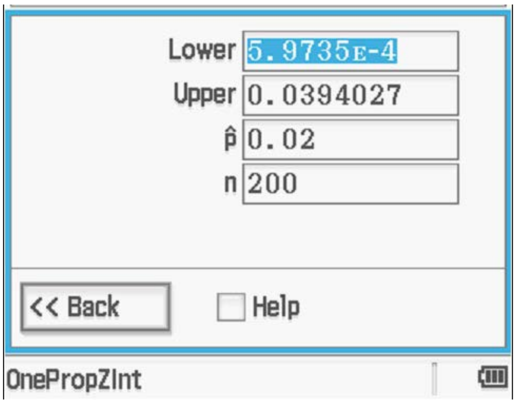

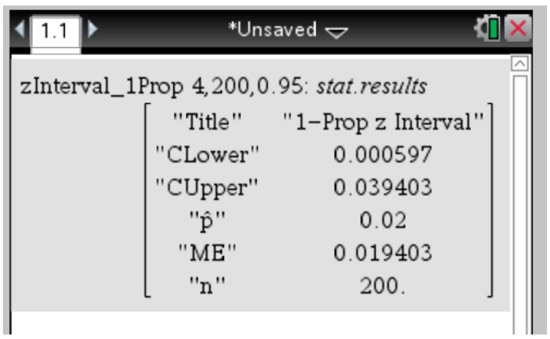

calculates correct sample proportion✓

calculates standard error correctly✓

calculates interval correctly✓

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d Given that $0.01 \in [0.0006, 0.0394]$, there is insufficient evidence to suggest that the historical data isn't relevant to current standards.

gives a reason based on the sample proportion lying in the confidence interval✓

e The lower end of the confidence interval is below 0.005, so the lower target is met. However, the higher end is above 0.01, so the upper target is not met.

refers to targets with reference to confidence interval✓

states decision✓

Question 7 [SCSA MM2021 Q11] (15 marks)

(✓ = 1 mark)

a $\hat{p} \sim N\left(0.23, \frac{0.23 \times 0.77}{400}\right)$

That is,

$\hat{p} \sim N\left(0.23, 0.00210^2\right)$

states the distribution is normal✓

gives the correct mean✓

gives the correct variance✓

b $P(\hat{p} < 0.20) = 0.07697$

writes correct probability statement✓

uses correct mean and standard deviation✓

obtains final answer✓

c $\hat{p} = \frac{55}{200} = 0.275$

calculates sample proportion correctly✓

d $E = 2.576\sqrt{\frac{0.275 \times 0.725}{200}} = 0.08133$

substitutes correct values in the formula for margin of error✓

calculates margin of error correctly✓

e

$$95\%\ \text{CI} = \left[0.275 - 1.960 \times \sqrt{\frac{0.275 \times 0.725}{200}}, 0.275 + 1.960 \times \sqrt{\frac{0.275 \times 0.725}{200}}\right]$$

$$95\%\ \text{CI} = [0.2131, 0.3369]$$

uses the correct critical value from the normal distribution✓

substitutes correct values in the expression for the confidence interval✓

calculates the confidence interval correctly✓

f The 95% confidence interval for the new sample (from part **e**) contains the value of the proportion for the earlier sample, so based on this we concluded that there is not enough evidence to determine whether the voters likely to vote for the Sustainable Energy party in this electorate has increased.

states that the confidence interval contains the proportion from the earlier sample✓

concludes that there is not enough evidence to determine whether the proportion has increased✓

g

1. Voters either vote for the party or not (success or failure).
2. The voters likely to vote for the Sustainable Energy party are independent of each other. This is a reasonable assumption.
3. The probability of a voter likely to vote for the Sustainable Energy party is the same for all voters. This is most likely not valid, as the probability may depend on other factors, such as the age of the voter, occupation, socio-economic status, employment status.

states the first assumption with justification✓

states the second assumption with justification✓

states the third assumption with justification✓

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